HYDROLOGIC OPTICS

Volume III. Solutions

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Pacific Marine Environmental Laboratory

It is the man
Not the method
That solves the problem

H. Maschke

Volume III

PART II THEORY OF LIGHT FIELDS

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Canonical Forms of the Equation of Transfer

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Natural Solutions of the Equation of Transfer

PREFACE

A brief survey of the methods of solution of radiative transfer equations* conducted recently showed the extremely wide variety of methods now available to modern researchers in this discipline. However, there are some methods which go to the very heart of the equation of transfer, notably the natural method of solution (via scattering order decomposition), and which stand foremost by virtue of their power and elegance. Another such method is the spherical harmonic method, which attempts to extend the time-honored technique of separation of variables to the equation of transfer. Finally there is the method of diffusion equations of both approximate and exact type. I have selected these three major methods for exposition here. The remaining principal method of solution, namely the invariant imbedding method, is reserved for study in Vols. IV and V.

As always, I have been concerned with the fundamental questions of the discipline, those that throw light on the conceptual structure of our subject. For this reason I have avoided discussing various extreme types of techniques of solution, chief among which are the abstract mathematical techniques concerned with uniqueness and existence questions, or with unrealizable algorithms which have no physical content and hence no role in the mathematical-physical foundations of the subject. Moreover, such techniques as the Monte Carlo method were avoided because of their zero conceptual content. Finally, I have not included purely numerical tabulations of solutions of the equation of transfer. Nothing is simpler in these days of powerful computers and exceedingly accomplished computer programs, to rack up several volumes of specialized solution tabulations for various selected geometries. I do not deny the utility of such tabulations; I am simply adhering to my originally imposed constraints which try to keep this (already extensive) work on the track of fundamental conceptual constructions, rather than numerical and experimental compilations.

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Honolulu, Hawaii
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*Preisendorfer, R. W. "A Survey of Theoretical Hydrologic Optics," J. Quant. Spectrosc. Radiat. Transfer *8*,325 (1968).

A bibliography of solution procedures may be found in: Lenoble, J. (editor) *Standard Procedures to Compute Atmospheric Radiative Transfer in a Scattering Atmosphere*. Report of the Radiation Commission of the International Association of Meteorology and Atmospheric Optics (International Union of Geodesy and Geophysics) Vols. I-IV (1974). Laboratoire d' Optique Atmospherique, Université des Sciences et Techniques de Lille, France.

CHAPTER 4

CANONICAL FORMS OF THE EQUATION OF TRANSFER

4.0 Introduction

In this chapter we begin a systematic construction of the main laws of radiative transfer theory by means of the principles of Chapter 3, with the particular goal in mind of deriving certain special types of transfer equations for the main radiometric concepts. These equations have been found most useful in the applications of the theory to the study of light in both the sea and the atmosphere. This task will occupy our attention during this and the following four chapters. In the present chapter our purpose is to obtain the *canonical equations* of transfer for radiance.

The sense in which we use the word "canonical" is twofold. First of all, "canonical" is to denote a fundamental well-established form of the equation of transfer--a form which has evolved and eventually gained universal acceptance over a two hundred year period of development. This is not to say that the canonical form of the equation of transfer is given first priority in every mathematical investigation of the transfer of radiant energy in optical media; rather, it is simply intended to signify the fact that the canonical form of the equation of transfer has been applied and independently rediscovered with sufficient frequency in various fundamental investigations in different subfields of radiative transfer over the years, that it has eventually taken on the role of an enduring useful landmark in the general theory. The second sense of the word "canonical" as used here is of a more technical nature; it is to denote the fact that the equations are written in a form of great simplicity without decreasing generality, and in a way that is independent of any particular coordinate system. Of the two senses, the first by far is to be considered the dominant sense in what follows.

The earliest recorded appearance of the canonical form of the equation of transfer was in the work of Bouguer, in whose classical treatise [28] appears a special but unmistakable form of the equation. This equation was unearthed and dusted off by Middleton in his studies of Bouguer's work. Specifically, Middleton observes [28] that: "Bouguer integrated the contributions of many elementary layers (dx) by a geometrical construction, and showed that [in modern notation] the apparent brightness of an object at distance x is

$$B(x) = ae^{-\alpha x} + b(1-e^{-\alpha x}) \quad ." \qquad (1)$$

The salient features of this equation, those that make it "canonical" in the technical sense, can be described in terms of the concepts developed in Chapter 3. First of all we observe that (1) has the Gestalt of (5) of Sec. 3.13, where the term $ae^{-\alpha x}$ corresponds to N_r^o in equation (5) of Sec. 3.13, the term $b(1-e^{-\alpha x})$ corresponds to N_r^*, and the term $B(x)$ to N_r. Thus $B(x)$ is interpretable as the apparent radiance of an object (Sec. 3.13) as seen over a path of length x, where the path radiance of the path is $b(1-e^{-\alpha x})$ and the inherent radiance of the object is a. The particular manner in which a, b, and $e^{-\alpha x}$ occur in the algebraic form of (1) characterize (1) as *canonical*. Equation (1) is substantially the algebraic form of $B(x)$ deduced by Bouguer from empirical observations. According to Middleton, however, Bouguer ostensibly missed the full physical significance of the terms a and b. Hindsight and a fully developed theory now let us view a and b in quite simple terms. Thus a in (1) is the inherent radiance of the object which is transmitted over the path with beam transmittance $e^{-\alpha x}$. Hence α must be the attenuation coefficient of the path (our α of Sec. 3.11). The term b is a simple instance of the general concept of equilibrium radiance which will be introduced and studied in detail in this chapter. Physically, b is the radiance of a very long uniformly lighted homogeneous path. Mathematically, b is simply the limit of $B(x)$ as $x\to\infty$. The radiance b is independent of location along the uniformly lighted homogeneous path, and in real life is closely approximated by the horizon radiance under suitable atmospheric conditions. The horizon radiance remains ostensibly constant, for example, on a transcontinental jet flight at 10,000 m altitude over large segments of the flight path. The observed horizon radiance seen by the jet pilot is the real counterpart to the equilibrium radiance b in (1). Of course similar interpretations of a,b and corresponding interpretations of (1) apply to horizontal lines of light in the sea, under suitable conditions.

In the present chapter we shall develop a hierarchy of canonical equations of transfer for radiance starting with the simplest of applied situations and concluding with what appears to be the most comprehensive canonical equation of transfer for physically meaningful contexts. Equation (1) will fall somewhere in the lower middle of this hierarchy, that is, somewhere in the neighborhood of the Koschmieder equation of Sec. 4.3. Throughout this chapter, unless specifically noted otherwise, all optical media will be considered emission-free, in the steady state, and of constant index of refraction. This condition does not constitute any significant loss of generality in terrestrial settings while permitting a simple exposition of the main idea of the canonical equation.

4.1 Radiance in Transparent Media

We take up first the simplest case in which the canonical equation of transfer can occur: transparent optical

media. A *transparent* optical medium X is one in which $\alpha(x,\xi) = 0$ and $\sigma(x;\xi';\xi) = 0$ for every x in X and ξ',ξ in Ξ. An example of a transparent optical medium is a block of glass which does not appreciably absorb or scatter radiant energy. Under these conditions, the integral equation of transfer (2) of Sec. 3.15 associated with a path $\mathcal{P}_r(x,\xi)$ in a vacuum takes the form:

$$\boxed{N(z,\xi) = N(x,\xi)} \quad . \tag{1}$$

Where $z = x + \xi r$. This instance of the equation of transfer is clearly interpretable also as an instance of the radiance invariance law(2) of Sec. 2.6.

In the case of a transparent optical medium in which the index of refraction varies with location along $\mathcal{P}_r(x,\xi)$, the n^2-law for radiance (4) of Sec. 2.6

$$N(z,\xi)/n^2(z) = N(x,\xi)/n^2(x) \tag{2}$$

governs the magnitude of $N(z,\xi)$ along $\mathcal{P}_r(x,\xi)$.

The preceding two laws also can be made to follow from the appropriate integrodifferential form of the equation of transfer. This would be equation II of Sec. 21 in Ref. [251], which in turn is deducible from the interaction principle. Thus we would deduce from this equation that

$$\boxed{\frac{d\ N(x,\xi)/n^2(x)}{dr} = 0} \quad , \tag{3}$$

from which follows (2). Equation (3) of Sec. 3.15 yields in particular:

$$\frac{dN(x,\xi)}{dr} = 0 \tag{4}$$

for the case of a transparent medium with constant index of refraction. From this follows (1). Clearly (4) is a special case of (3), so that (3) may be considered the basic equation for radiative transfer in transparent media.

4.2 Radiance in Absorbing Media

The next simplest case of an optical medium containing a radiative transfer process is that of a purely absorbing medium. A *purely absorbing* optical medium X is one in which $\sigma(x;\xi',\xi) = 0$ for every x in X and ξ',ξ in Ξ. An everyday example of a purely absorbing medium is a uniformly exposed photographic negative. By holding such a negative to the eye and viewing one's surroundings through it, the principal radiative transfer feature of a purely absorbing medium is readily perceived: Such media characteristically *decrease* the radiance of a scene by a factor which depends only on the inherent optical and geometric makeup of the medium and which does not depend on the surrounding light field. If the absorption properties of an optical medium X are uniform throughout

X, then the factor of the observed decrease is a simple exponential factor exp $\{-\alpha r\}$ depending only on the attenuation coefficient α and the length r of one's path of sight through the medium. In particular no light from the surrounds of the path will be added to that of the path. Indeed, if the universe were made up only of absorbing material, radiative transfer theory beyond the use of the exponential function would not exist, so simple and straightforward is the form of (1) of Sec. 3.15 when reduced to pure absorption case:

$$\boxed{N_r(z,\xi) = N_o(x,\xi)\, T_r(x,\xi)} \tag{1}$$

Equivalently (3) of Sec. 3.15 reduces to:

$$\frac{dN(z,\xi)}{dr} = -\alpha(z,\xi)N(z,\xi) \tag{2}$$

However, in all real media, absorption mechanisms are accompanied by scattering mechanisms in the radiative processes within such media. Hence, the losses summarized by the volume attenuation function α include scattering losses in addition to the absorption losses. The losses due to scattering at a typical point of a path $\mathcal{P}_r(x,\xi)$ in general optical medium X are readily characterized using the volume scattering function of Sec. 3.14. Indeed the integral:

$$\int_{\Xi} \sigma(z;\xi;\xi')\, d\Omega(\xi')$$

represents the total radiance loss by a beam of given wavelength and unit radiance, under scattering without change in wavelength (elastic scatter) and per unit length at z, along the direction ξ of the path $\mathcal{P}_r(x,\xi)$ at that point. This interpretation follows readily from the developments in Sec. 3.14.

Let us write:

$$\text{"}s(z,\xi)\text{"} \quad \text{for} \quad \int_{\Xi} \sigma(z;\xi;\xi')\, d\Omega(\xi') \tag{3}$$

We call s the *volume total scattering function on X.* Further, let us write:

$$\text{"}a(z,\xi)\text{"} \quad \text{for} \quad \alpha(z,\xi) - s(z,\xi) \tag{4}$$

so that:

$$\alpha(z,\xi) = a(z,\xi) + s(z,\xi)$$

We call the function a which assigns to each point z on $\mathcal{P}_r(x,\xi)$ the value $a(z,\xi)$, the *volume absorption function* on X. The interpretation of $a(z,\xi)$ is straightforward: it represents the loss of radiance per unit length at point z

on $\mathcal{P}_r(x,\xi)$ of a beam of unit radiance, the loss being due to two physical mechanisms: (i) the scattering of some of the incident radiant flux into radiant flux of a different wavelength than that of the incident beam (*inelastic scatter or transpectral scatter*); (ii) the conversion of some of the incident radiant flux into non-radiant energy (*true absorption*). Some forms of non-radiant energy pertinent here are: the potential energy of higher stationary states in atomic systems, and the kinetic energy of motion of the molecules of the optical medium. Since $\alpha(z,\xi)$ represents losses due to all the mechanisms namely elastic scatter, inelastic scatter, and true absorption, we expect on physical grounds that $a(z,\xi)$ is nonnegative for every z and ξ in its domain of definition, and we hypothesize the appropriate inequality to hold henceforth between α and s so that this nonnegativity of $a(z,\xi)$ is the case.

It is worthwhile to bring explicity to the reader's attention the particular role played by the volume absorption function in radiative transfer theory. The function plays the role of a catchall of all radiant flux losses undergone by a beam of radiant flux other than by the mechanism of elastic scatter. The two fundamental (or primary) optical properties of a medium X are α and σ. The concept a as defined in (4) is a secondary property, that is, one that is derived from α and σ as shown. The secondary nature of the concept a follows from the fact that in practice absorption cannot be observed directly, but only indirectly by means of monitoring the initial and final states of a beam in transmission and scattering arrangements in experimental settings.

Using the definition (4) of the function a, we can write (1) or (2) in the form:

$$N_r(z,\xi) = N_o(x,\xi) \exp\left\{ - \int_o^r a(x',\xi)\,dr' \right\}$$

$$\frac{dN(z,\xi)}{dr} = -a(x,\xi)\, N(x,\xi) \tag{5}$$

where the integration is along the path $\mathcal{P}_r(x,\xi)$ with $z = x + r\xi$ (see Fig. 3.33).

4.3 Koschmieder's Equation for Radiance

A classical problem of radiative transfer theory in either the atmosphere or in the sea is to determine the apparent radiance of an object as seen along a path of sight $\mathcal{P}_r(x,\xi)$ which lies in a homogeneous and uniformly lighted region of an optical medium. Specifically, the problem is to determine the apparent radiance $N_r(z,\xi)$ given α and σ along $\mathcal{P}_r(x,\xi)$, and $N_o(x,\xi)$ at the initial endpoint x of the path, along with the fact that each point of $\mathcal{P}_r(x,\xi)$ is irradiated by the same radiance distribution (which may, however, depend arbitrarily on ξ'). This situation (or some reasonable approximation of it) arises often in the atmosphere and the sea, notably along horizontal paths of sight,

and the reader should be able to cite many personally observed instances of it. Koschmieder studied this classical setting in detail, and in 1924 published in [141] his analytic expression for $N_r(z,\xi)$ which was derived after lengthy preliminaries and under the radiometric conditions stipulated above. We turn now to a modern derivation of the expression for $N_r(z,\xi)$.

Returning to (1) of Sec. 3.15 we assume α and σ are independent of z along $\mathcal{P}_r(x,\xi)$. Then:

$$T_r(x,\xi) = e^{-\alpha r}$$

where "α" denotes the assumed fixed value of the volume attenuation function along $\mathcal{P}_r(x,\xi)$. Furthermore, since the radiance distribution $N(z,\cdot)$ is independent of z along the path then $N_*(z,\xi)$ is also independent of z along the path and we shall abbreviate this fixed value by "N_*". Equation (1) of Sec. 3.15 then reduces to:

$$N_r(z,\xi) = N_o(x,\xi)e^{-\alpha r} + N_* \int_o^r e^{-\alpha(r-r')}dr'$$

and with the abbreviations "N_r" for $N_r(z,\xi)$ and "N_o" for $N_o(x,\xi)$, this simplifies immediately to:

$$\boxed{N_r = N_o e^{-\alpha r} + N_q(1-e^{-\alpha r})} \qquad (1)$$

where we have written:

$$\text{"}N_q\text{"} \quad \text{for} \quad N_*/\alpha \qquad (2)$$

Equation (1) is *Koschmieder's equation* which relates apparent radiance N_r to N_o on a path $\mathcal{P}_r$ in an optical medium along which α and σ are constant valued and along which the value N_* of the path function is constant. The radiance N_q is called the *equilibrium radiance* for $\mathcal{P}_r$. The significance of N_q is seen by letting $r\to\infty$ in (1), or alternately by contemplating the integrodifferential equation for N_r associated with $\mathcal{P}_r$ as given in (3) of Sec. 3.15:

$$\frac{dN_r}{dr} = -\alpha N_r + N_* \qquad (3)$$

Under our present assumptions, (3) is a relatively innocuous first order differential equation in which α and N_* are constants and N_r is the unknown function. Using (2) we can rewrite (3) as:

$$\frac{dN_r}{dr} = \alpha(N_q - N_r) \qquad (4)$$

from which we can immediately read the physical significance of N_q: If $N_r < N_q$ at a point on the path, then $dN_r/dr > 0$, i.e., N_r is increasing at that point. In general, N_r always tends toward the fixed radiance N_q, and $dN_r/dr = 0$ if and only if $N_r = N_q$. Therefore N_q takes on the aspect of an equilibrium value (in an every day sense) toward which the values N_r unceasingly tend. The equilibrium radiance N_q is often observable over long horizontal uniformly lighted paths through a homogeneous natural aerosol or hydrosol.

It should be observed that the derivation of (1) places no conditions on the orientation or the location of the path $\mathcal{P}_r$ in an optical medium. The essential point to observe in the derivation is that (1) follows from (1) of Sec. 3.15 upon assuming only that α, σ and N_* are constant long $\mathcal{P}_r$. This leaves $\mathcal{P}_r$ free to be vertical, inclined, or horizontal, as the case may be. An interesting example of (1) for inclined paths of sight in the atmosphere may be obtained from the results in [71].

4.4 The Classical Canonical Equation

In this section we continue to ascend the ladder of generality and derive still further instances of canonical radiance equations. We still have not reached the most general physical setting in which the canonical equation can hold, but we have reached the point where the full canonical structure of the equation finally emerges, and we turn now to the derivation of that form.

Let $\mathcal{P}_r(x,\xi)$ be an arbitrary line of sight in a homogeneous optical medium X. To fix ideas, let the medium X be a natural hydrosol, and let us adopt the standard coordinate frame for such a setting (Sec. 2.4 and Fig. 2.3). Let $\mathcal{P}_r(x,\xi)$ be positioned as shown in Fig. 4.1.

With the geometry fixed as in Fig. 4.1, we now assume α and σ to be independent of location along the generally inclined path $\mathcal{P}_r(x,\xi)$, and that the light field does not vary over a given horizontal plane, i.e., the light field is stratified. The new feature of the canonical equation appears by assuming that there exists a nonnegative real number K (which is less than α) such that:

$$N_*(z,\xi) = N_*(z_0,\xi)\, e^{-K(z-z_0)} \tag{1}$$

for every path $\mathcal{P}_r(x,\xi)$ in X. This means that we are hypothesizing an exponential decrease of $N_*(z,\xi)$ with depth z in X. The justification for this assumption rests on both experimental and theoretical grounds. For an experimental justification, see Sec. 1.2; for theoretical justifications see Secs. 1.3, 7.10, 8.5, 8.6 and Sec. 10.7. For the present, we are concerned primarily with the resultant form of (6) of Sec. 3.13 to which this assumption leads us. Thus starting with (6) of Sec. 3.13, we have:

$$N_r(z,\xi) = N_0(x,\xi)T_r(x,\xi) + \int_0^r N_*(x',\xi)T_{r-r'}(x',\xi)\, dr'$$

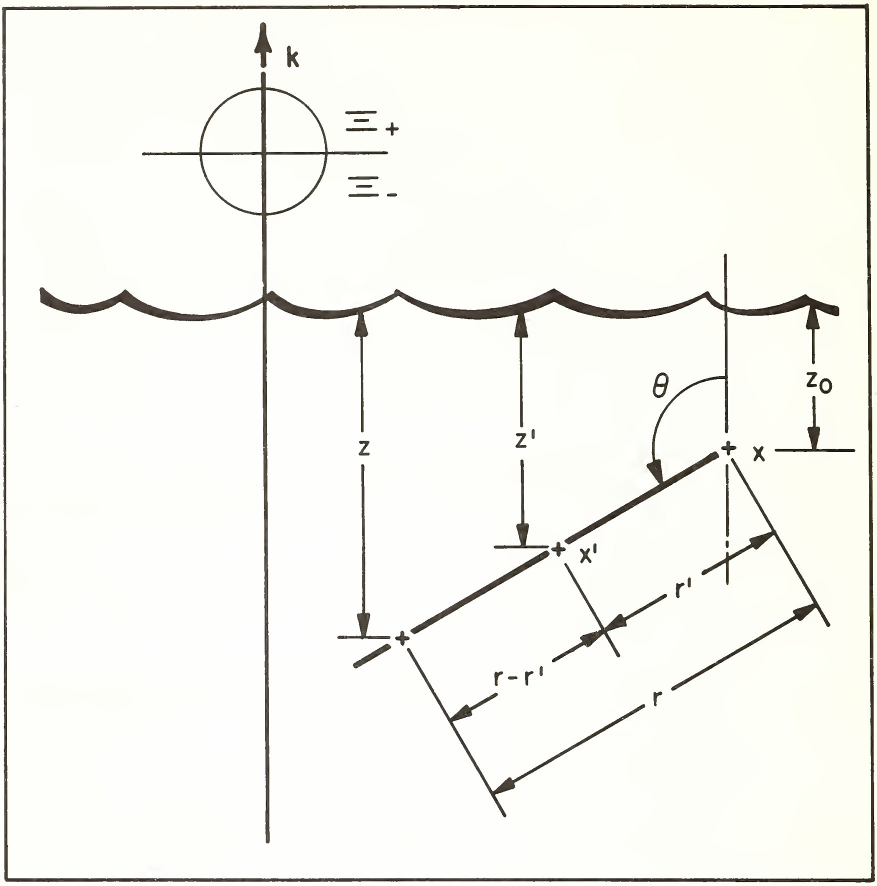

FIG. 4.1 Setting for the derivation of the classical canonical equation for radiance.

Since $N_*(x',\xi)$ depends only on the depth z' of the point x' along $\mathcal{P}_r(x,\xi)$, we may drop references to x and y coordinates and need only relate the variable of integration r' with z' using the relation:

$$z' = z_o - r' \cos \theta$$

so that:

$$dz' = - \cos \theta \; dr' \quad .$$

The equation for $N_r(z,\xi)$ with "z" denoting depth, then becomes:

$$N_r(z,\xi) = N_o(z_o,\xi)e^{-\alpha r} + \int_0^r N_*(z_o,\xi)e^{-K(z'-z_o)}e^{-\alpha(r-r')} \, dr'$$

That is:

$$N_r(z,\xi) = N_o(z_o,\xi)e^{-\alpha r} + N_*(z_o,\xi)\int_0^r \exp\{Kr' \cos\theta - \alpha(r-r')\}dr'$$

$$= N_o(z_o,\xi)e^{-\alpha r} + N_*(z_o,\xi)e^{-\alpha r}\int_0^r \exp\{(\alpha + K\cos\theta)r'\}\, dr' \ .$$

Hence:

$$N_r(z,\xi) = N_o(z_o,\xi)e^{-\alpha r} + \frac{N_*(z_o,\xi)^{-\alpha r}}{\alpha + K\cos\theta}\left[\exp\{(\alpha + K\cos\theta)r\} - 1\right] \quad .$$

Using (1) once again and the connection between z and r along $\mathcal{P}_r(x,\xi)$, we have:

$$\boxed{N_r(z,\xi) = N_o(z,\xi)e^{-\alpha r} + \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\left[1 - e^{-(\alpha + K\cos\theta)r}\right]} \qquad (2)$$

which is the desired form of the *classical canonical equation for $N_r(z,\xi)$.*

We now make several observations on the structure of (2). First of all, (2) is a proper generalization of equations (1) of Secs. 4.1 and 4.2, and of Koschmieder's equation in Sec. 4.3, reducing to the latter either when $K \neq 0$ and $\theta = \pi/2$, or when $K = 0$ and θ arbitrary. In all real natural hydrosols, $K \neq 0$ so that Koschmieder's equation holds in natural hydrosols only when $\theta = \pi/2$. In the atmosphere on relatively clear days, $K = 0$ (very nearly) over relatively long horizontal or inclined paths, and so Koschmieder's equation holds over relatively extensive regions in the atmosphere (cf., Ref. [71]).

As a second observation, we note that the main use of (2) is to predict the apparent radiance N_r of given objects in natural optical media when α, K and N_o are known or estimable. Furthermore, (2) yields a useful estimate of the path radiance N_r^* generated over a path of sight in an optical medium, that is,

$$\boxed{N_r^*(z,\xi) = \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\left[1 - e^{-(\alpha + K\cos\theta)r}\right]} \quad . \qquad (3)$$

If we now write:

$$\text{"}N_q(z,\xi)\text{" for } N_*(z,\xi)/\alpha(z,\xi) \quad , \qquad (4)$$

which is a straightforward generalization of the equilibrium radiance defined in (2) of 4.3, (3) may then be rendered in the form:

$$N_r^*(z,\xi) = \frac{N_q(z,\xi)}{1 + \frac{K}{\alpha}\cos\theta}\left(1 - e^{-(\alpha + K\cos\theta)r}\right) \qquad (5)$$

This shows that the equilibrium radiance $N_q(z,\xi)$ is observable only for infinitely long *horizontal* paths in natural hydrosols. For other paths, N_q contributes to the observable quantity N_r^* in the manner shown in (5) but itself is not directly observable.

As a third observation imagine a descent into a deep hydrosol, such as a deep lake or part of the ocean. Let $N_o(z,\xi)$ be the inherent radiance of the air-water boundary for directions ξ in Ξ_-, and $N_o(z_1,\xi)$ be the inherent radiance of the lower boundary of the medium for directions ξ in Ξ_+. Then when the optical distance αr to the boundaries becomes relatively large, $e^{-\alpha r}$ becomes relatively small. Under such conditions $N_r(z,\xi)$ is expressed essentially in the form (5), with the exponential term in (5) also negligible. Hence, at relatively great depths in deep natural hydrosols we have essentially:

$$N(z,\xi) = \frac{N_q(z,\xi)}{1 + \frac{K}{\alpha}\cos\theta} \qquad (6)$$

where "r" has now been dropped from the notation as being inessential. Thus the radiance distribution $N(z,\xi)$ at relatively great depths z is basically an ellipsoid of revolution with vertical axis and with eccentricity $\varepsilon = K/\alpha$, which is modified, as shown in (6), by the equilibrium radiance distribution $N_q(z,\xi)$ at the same depth.

There is a special class of homogeneous optical media for which (6) reduces to precisely the ellipsoid of revolution of eccentricity ε, namely media for which $\sigma(z,\xi';\xi)$ is independent of ξ' and ξ. For such media we have from the definition (3) of Sec. 4.2:

$$\sigma(z,\xi',\xi) = s/4\pi \quad , \qquad (7)$$

so that from (8) of Sec. 3.14:

$$N_*(z,\xi) = s\,h(z)/4\pi \quad , \qquad (8)$$

where h(z) is the scalar irradiance induced by $N(z,\xi)$ (Sec. 2.7).

If we write:

"ρ" for s/α ,

which is the *albedo for single scattering, or scattering-attenuation ratio,* then (6) becomes:

$$N(z,\xi) = \frac{\rho h(z)}{4\pi(1 + \frac{K}{\alpha}\cos\theta)} \qquad (9)$$

It is quite clear from (8) that $N_*(z,\xi)$ is independent of ξ, and that:

$$h(z) = h(z_0)e^{-K(z-z_0)} \tag{10}$$

From this we see that there is in $N(z,\xi)$ a multiplicative uncoupling of depth (z) and directional (θ or ξ) parameters and that scalar irradiance and path function values both decrease exponentially with depth and at equal rates. This multiplicative uncoupling of z and ξ can be represented as a product of a function of z only and a function of ξ only; it is of far-reaching importance in the general theory of solutions of the equation of transfer. (See Sec. 6.6.) Furthermore, we shall return to (6) and to (9) once again in Sec. 10.5, when the problem of the asymptotic radiance distribution at great depths is examined in a more rigorous fashion.

The preceding observations point up the versatility of the classical canonical form of the equation of transfer and suggest that of all the various equations encountered in practice, (2) is perhaps the most handy and succinct rule of thumb on natural light field behavior to carry around in one's memory. To add to the evidence of the utility of (2) we now deduce from it two further features of natural light fields.

First, we may ask: What is the behavior of path radi- $N_r^*(z,\xi)$ for very short paths of sight? This question directs attention to a situation which complements that centered around (6). Now from elementary calculus it is at once clear that:

$$1-e^{-(\alpha + K \cos \theta)r} = (\alpha + K \cos \theta)r + o(r)$$

where o(r) is a function such $\lim_{r\to 0} o(r)/r = 0$, so that for small r, o(r) is an infinitesimal of order higher than r. Therefore (3) reduces, within first order terms in r, to:

$$\boxed{N_r^*(z,\xi) = N_*(z,\xi)r} \tag{11}$$

Hence the answer to the question posed above is that for short paths of light $N_r^*(z,\xi)$ varies linearly with r, the proportionality factor being $N_*(z,\xi)$.

Finally, we may ask: What is the structure of the apparent radiance distribution near the air-water boundary, i.e., for very shallow depths? This query rounds out the complementary situation to that in (6) which describes the light field at relatively great depths. We take a simple case to illustrate the manner in which such questions may be answered using the canonical equation for radiance. Suppose the sky above the natural hydrosol is a deep blue and the sun is the only bright source of light in the azure hemisphere. Let attention be directed at a relatively dark point of sky away from the sun's disc. Hence the radiance $N_0(0,\xi)$ (with ξ in Ξ_-) from that portion of the sky as seen from just below the surface is very small compared to the sun's radiance.

Now keeping ξ fixed, let depth z increase. If the term $N_o(0,\xi)e^{-\alpha r}$ is negligible, as we now wish it to be, then $N_r(z,\xi)$ is given essentially by $N_r^*(z,\xi)$. For small depths z (and hence small path lengths r), $N_r^*(z,\xi)$ is essentially 0. As z increases through small depths, $N_r^*(z,\xi)$ builds up linearly in magnitude according to (11). For still further increases in z, $N_r^*(z,\xi)$ eventually levels off, reaches a maximum, and then subsequently plunges toward zero exponentially with rate K as $z\to\infty$. All this information is read off during an inspection of (3). We can obtain an estimate of the depth z at which the maximum path radiance is reached. Thus, from elementary calculus we find the maximum of $N_r^*(z,\xi)$ with respect to z by holding ξ fixed, differentiating with respect for z, and setting the derivative equal to zero. First, recalling that:

$$\frac{dr}{dz} = -\sec\theta \quad ,$$

we then use (3) to differentiate $N_r^*(z,\xi)$:

$$\begin{aligned}
\frac{dN_r^*(z,\xi)}{dz} &= \frac{dN_*(z,\xi)/dz}{\alpha + K\cos\theta}\left[1-e^{-(\alpha + K\cos\theta)r}\right] + \\
&+ \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\frac{d}{dz}\left[1-e^{-(\alpha + K\cos\theta)r}\right] \\
&= \frac{-KN_*(z,\xi)}{\alpha + K\cos\theta}\left[1-e^{-(\alpha + K\cos\theta)r}\right] \\
&+ \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\cdot(\alpha + K\cos\theta)e^{-(\alpha + K\cos\theta)r}\cdot\frac{dr}{dz} \quad .
\end{aligned}$$

Setting the derivative to zero, and solving for z_m, the value of z which maximizes $N_r^*(z,\xi)$, we have:

$$\boxed{z_m = \frac{-\ln(-\varepsilon/\sec\theta)}{\alpha(\varepsilon + \sec\theta)}} \tag{12}$$

where "ε" is again written for K/α.

Still further, more realistic models can be constructed for the radiance patterns at shallow depths in natural waters using similar procedures but now based on the full form of the classical canonical equation (2); the explorations of such models and that of (12) are still in their early stages of development and are left to interested students of the subject. Figure 4.2, taken from [298], depicts a comparison plot between some computed values of $N_r(z,\xi)$ (solid curve) using (2) with actual observed radiances and path function values at the surface obtained in a real situation, and thereby illustrates graphically the predictive power of the simple model of natural light fields summarized in (2). Observe in particular the reasonably good agreement between the predicted and observed value of the depth z_m at which maximum radiance occurs.

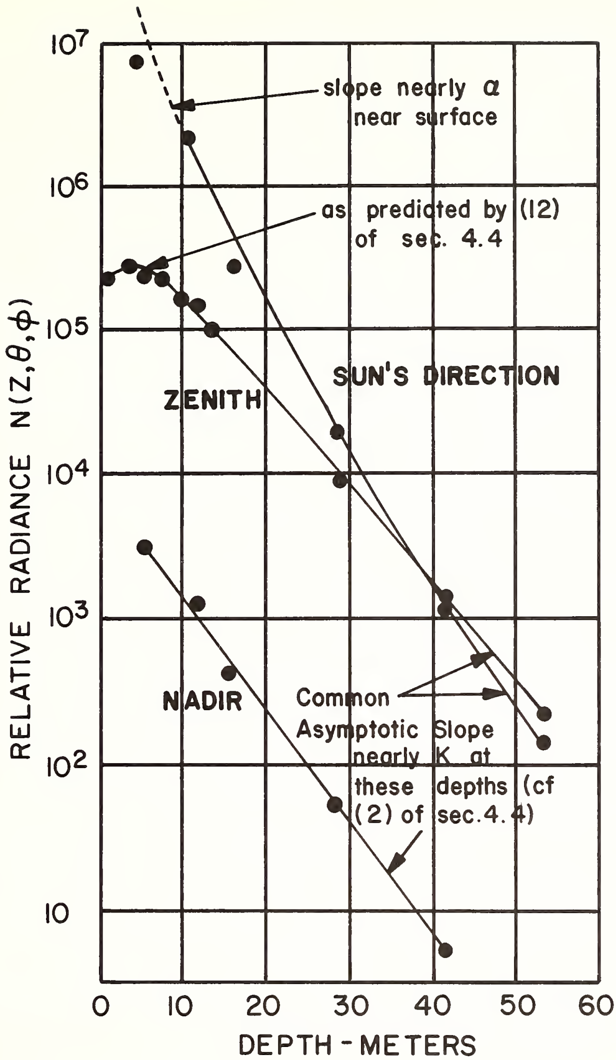

FIG. 4.2 Tyler's experimental verification (dots) of the curves predicted by the classical canonical equation for radiance.

Further models beyond (2) may now be explored by letting K vary in various known ways with depth, so that slightly more general forms than (1) are the starting points for the integration of the equation of transfer. In view of the fact that N_* generally behaves very nearly in an exponential manner with depth, these departures of K from constancy need only be very slight to cover most real situations. The basis for these generalizations is given in (19) of Sec. 4.5.

4.5 The General Canonical Equation for Radiance

The purpose of this section is to draw attention to a general pattern discernible in the various expressions, derived in the foregoing sections, for the apparent radiance N_r which is the logical common denominator of the large collection of analytic expressions for N_r which occur in the everyday studies of atmospheric and hydrologic optics. No specific or general problems of applied radiative transfer theory are intended to be solved for the moment, and no new numerical methods are expected to be immediately forthcoming. We seek instead to determine a general equation which will unify and hold within its form, as special cases, the various ways of correctly representing the apparent radiances of both near and distance parts of one's radiometric environment. In short, we extract from the examples discussed above and others in the literature, the *general canonical representation* of the apparent radiance function which will hold for all cases occurring in geophysical optics.

The key concept leading to the formulation of the appropriate canonical representation of apparent radiance turns out to be that of a generalized form of *radiance transmittance* associated with a path of sight $\mathcal{P}_r(x,\xi)$ in an optical medium. This concept is suggested after a study of the integral representation of the beam transmittance $T_r(x,\xi)$ associated with $\mathcal{P}_r(x,\xi)$, as given in (3) of Sec. 3.11, while keeping in mind the basic property of $T_r(x,\xi)$ as summarized in (4) of Sec. 3.10, that is, the fact that $T_r(x,\xi)$ is the ratio of the beam-transmitted radiance $N_r^o(z,\xi)$ to the initial inherent radiance $N_o(x,\xi)$ over a path $\mathcal{P}_r(x,\xi)$. Suppose now we take the ratio of $N_r(z,\xi)$ to $N_o(x,\xi)$, i.e., of the apparent radiance to the inherent radiance over the path $\mathcal{P}_r(x,\xi)$. Let us call this ratio the *radiance transmittance* of the path $\mathcal{P}_r(x,\xi)$. It is quite evident that the beam transmittance and the radiance transmittance of a given path are generally two distinct numbers. We now ask: Can the radiance transmittance just defined be given an integral representation analogous to that for beam transmittance? For, if so, then it is quite a simple matter to construct the appropriate generalization of (2) of Sec. 4.4 without the encumbrance of special restrictive assumptions of the kind in (1) of Sec. 4.4 which, while justifiable in many useful contexts, distract from the mathematical elegance and physical completeness of the resultant canonical representation of $N_r(z,\xi)$.

The requisite integral representation of the radiance transmittance is readily obtained by building an analogy on the fact that α, the key function in the integral representation of $T_r(z,\xi)$, is the logarithmic derivative of $N_r^o(z,\xi)$ along the path. This observation is based on (3) of Sec. 3.10 and (2) of Sec. 3.11. Some preliminary experimentation leads to the following definition of the appropriate analogue of α required in the present discussion. Thus let us write:

$$\boxed{\text{"}\mathbf{K}\text{" for } -\nabla N/N} \quad . \tag{1}$$

Here ∇ is the spatial gradient operator and N is a general radiance function defined and differentiable in a region

X such that N does not vanish in X. If $\mathcal{P}_r(x,\xi)$ is a path in X, and N_0 and N_r are the radiances along the path at points x and $x + r\xi$, respectively, then it is a simple exercise in calculus to show that, under the preceding conditions on N,

$$\boxed{N_r/N_0 = \exp\left\{-\int_0^r \xi \cdot \mathbf{K}\, dr'\right\}} \tag{2}$$

where the integration is along the path $\mathcal{P}_r(x,\xi)$. We shall call $\mathbf{K}$ the *general K-function* for radiance; it is a most useful concept not only in the present discussion, but in many practical investigations of light in natural media. By means of $\mathbf{K}$ we can cast the equation of transfer (3) of Sec. 3.15 into *canonical form* as follows: Since d/dr is the direction derivative operation along the path,

$$\frac{dN(z,\xi)}{dr} = \xi \cdot \nabla N(z,\xi) \quad , \tag{3}$$

and we have:

$$\xi \cdot \nabla N(z,\xi) = -\ \xi \cdot \mathbf{K}(z,\xi) N(z,\xi) \tag{4}$$

by (1). From this we see that an immediate effect of the introduction of $\mathbf{K}$ is to replace the differential operation occurring in the equation of transfer by an ostensibly algebraic operation. The effect of this replacement on the equation of transfer is striking, as may be seen by writing (3) of Sec. 3.15 in abbreviated form:

$$\xi \cdot \nabla N = -\ \alpha N + N_*$$

and using (1), the equation becomes:

$$-\ \xi \cdot \mathbf{K} N = -\ \alpha N + N_*$$

which, upon solving for N, becomes:

$$N = \frac{N_*}{\alpha - \xi \cdot \mathbf{K}}$$

or in more detailed notations:

$$\boxed{N(z,\xi) = \frac{N_*(z,\xi)}{\alpha(z,\xi) - \xi \cdot \mathbf{K}(z,\xi)}} \tag{5}$$

Equation (5) is the *general canonical form of the equation of transfer*. It forms a key step in the derivations of the present section, and will also be used later in our studies of optical properties of natural hydrosols (Sec. 9.5). But for the present the reader should compare (5) above with (6) of 4.4 and note the close resemblance between that earlier approximate formula and the present exact formula (5).

Canonical Representation of Apparent Radiance

We can turn now to the details of the derivation of the requisite canonical representation of apparent radiance. The derivation will be facilitated if we adopt the following notation. We write:

$$\text{"}T_r[f]\text{"} \quad \text{for} \quad \exp\left\{\int_0^r f\,dr'\right\} \tag{6}$$

for every admissible function f on $\mathcal{P}_r(x,\xi)$, i.e., for every f defined and integrable, over a path $\mathcal{P}_r(x,\xi)$ of an optical medium X. In this notation, beam transmittance becomes:

$$T_r(x,\xi) = T_r[-\alpha] \tag{7}$$

and radiance transmittance becomes $T_r[-\xi\cdot\mathbf{K}]$. Observe that if f and g are two admissible functions on $\mathcal{P}_r(x,\xi)$, then:

$$\left.\begin{aligned} &T_r[f+g] = T_r[f]T_r[g] \\ &\text{and that:} \\ &(T_r[f])^{-1} \equiv T_r^{-1}[f] = T_r[-f] \quad . \end{aligned}\right\} \tag{8}$$

Henceforth we shall assume that α and $\xi\cdot\mathbf{K}$ are admissible on each path.

We begin the general derivation with (5) of Sec. 3.13:

$$N_r = N_r^o + N_r^* \tag{9}$$

which is the general representation of apparent radiance in decomposed form, i.e., in terms of beam transmitted inherent radiance N_r^o and path radiance N_r^* on a path $\mathcal{P}_r(x,\xi)$.

We use (9) to suggest the construction of the following identity:

$$N_r = N_r^o + \left[N_r - N_r^o\right] \tag{10}$$

which of course has no physical content, and is logically equivalent to the statement:

$$0 = 0 \quad . \tag{11}$$

However, we next observe that:

$$N_r^o = N_o T_r[-\alpha] \tag{12}$$

and that:

$$N_r = N_o T_r[-\xi \cdot \mathbf{K}] \tag{13}$$

and with these observations, (10) is transformed with the help of (8) into:

$$\begin{aligned} N_r &= N_o T_r[-\alpha] + N_r\left(1 - T_r[-\alpha] T_r^{-1}[-\xi \cdot \mathbf{K}]\right) \\ &= N_o T_r[-\alpha] + N_r\left(1 - T_r[-(\alpha - \xi \cdot \mathbf{K})]\right) \end{aligned} \tag{14}$$

Even though (14) is entirely devoid of physical meaning, and even though it is logically equivalent to (11), it nevertheless seems to be on the verge of saying something *physically* significant by virtue of the fact that it has the general form of (2) of Sec. 4.4. At this point the canonical form (5) of the equation of transfer makes its entrance. By using (5) to replace N_r on the right side of (14), life is breathed, so to speak, into the cold symbolic clay of (14) and we obtain:

$$\boxed{N_r = N_o T_r[-\alpha] + \frac{N_*}{\alpha \ \xi \cdot \mathbf{K}}\left(1 - T_r[-(\alpha - \xi \cdot \mathbf{K})]\right)} \tag{15}$$

This is the desired *general form of the canonical representation of apparent radiance* N_r over a path $\mathcal{P}_r(x,\xi)$. The radiance N_r in (15) is no longer arbitrary and free as in (14); now N_r in (15) is indissolubly locked to the optical properties of the medium via the equation of transfer. Equation (15) is the most general form of (2) of Sec. 4.4 attainable for unpolarized steady radiance functions in a general source-free optical medium. The quantity $T_r[-\xi \cdot \mathbf{K}]$ in (13) is called the *radiance transmittance* associated with $\mathcal{P}_r(x,\xi)$. It will be studied further, along with related transmittance concepts, in Sec. 9.5.

The Canonical Form for Stratified Media

As an application of (15) we now derive the appropriate instance of the equation in an arbitrary stratified natural hydrosol. The result will be a canonical representation for N_r about midway in generality between (2) of Sec. 4.4 and (15) above. We shall use without further explanation the terrestrially based coordinate system for hydrologic optics described in Sec. 2.4. (See Fig. 4.1.)

The reduction of (15) begins with the observation that from (1) we have in general:

$$\mathbf{K} = \mathbf{i}I + \mathbf{j}J + \mathbf{k}K \tag{16}$$

where we have written:

$$\text{"I"} \quad \text{for} \quad -\frac{1}{N}\frac{\partial N}{\partial x}$$

$$\text{"J"} \quad \text{for} \quad -\frac{1}{N}\frac{\partial N}{\partial y}$$

$$\text{"K"} \quad \text{for} \quad -\frac{1}{N}\frac{\partial N}{\partial z}$$

and where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors for a right-hand Cartesian coordinate system. For the particular coordinate system of hydrologic optics (Fig. 4.1) we must replace (16) by:

$$\mathbf{K} = \mathbf{i}I + \mathbf{j}J - \mathbf{k}K \tag{17}$$

For a stratified plane parallel medium all radiometric and optical functions are independent of x and y. Hence the x and y derivatives I and J above are zero, and so:

$$\xi \cdot \mathbf{K} = -\xi\cdot\mathbf{k}K = -K\cos\theta \tag{18}$$

and (15) becomes

$$\boxed{N_r = N_o T_r[-\alpha] + \frac{N_*}{\alpha + K\cos\theta}\left(1 - T_r[-(\alpha + K\cos\theta)]\right)} \tag{19}$$

This equation is exact and completely general for plane parallel media; α and K have general depth and direction dependence. Other than the stratification condition summarized in (18) and the current choice of coordinates summarized in (17), the canonical equation (19) holds for completely arbitrary lighting conditions and optical properties in a plane-parallel optical medium. In particular it should be noted that the function K in (19) may, according to (1) and (17), be defined within the plane-parallel context directly by writing:

$$\text{"}K(z,\xi)\text{"} \quad \text{for} \quad \frac{-1}{N(z,\xi)}\frac{dN(z,\xi)}{dz} \quad . \tag{20}$$

This is an operational definition of $K(z,\xi)$ using directly observable radiances $N(z,\xi)$; and so K, as it occurs in (19), is quite general in the plane-parallel setting. We shall study the depth behavior of $K(z,\xi)$ in some detail in Secs. 10.5 and 10.6. The reader should particularly note that (20) may serve as an *operational* definition of K in stratified plane parallel media. The associated canonical form of the equation of transfer is:

$$\boxed{N(z,\xi) = \frac{N_*(z,\xi)}{\alpha(x) + K(z,\xi)\cos\theta}} \tag{21}$$

Equation (19) reduces to (2) of Sec. 4.4 upon requiring α and K to be independent of depth z in the hydrosol. For then:

$$T_r[-\alpha] = \exp\{-\alpha r\}$$

$$T_r[-(\alpha + K\cos\theta)] = \exp\{-(\alpha + K\cos\theta)r\}$$

This points up one of the primary reasons for using the logarithmic derivative in (1) for the definition of $\mathbf{K}$. In most natural hydrosols all radiometric quantities (radiance, path function, irradiance, etc.) have potentially constant logarithmic derivatives with respect to depth. Indeed, in Secs. 7.9, 7.10, and 7.11, it will be shown that this fact holds for quite wide geometrical and physical settings. This observation suggests further models of natural light fields that may be derived from (19). For by postulating a certain depth dependence of K suggested by experiment or theory (these are usually relatively mild dependences) and placing that depth dependence in (19), new models of N_r and N_r^* can be obtained which will fall somewhere between (2) of Sec. 4.4 and (19) as regards tractability in computation and fidelity of description of light fields.

4.6 Canonical Representation of Polarized Radiance

In this section we shall extend the notion of the canonical representation of apparent radiance to the polarized context. One consequence will be a representation of polarized radiance distributions in stratified natural hydrosols comparable in simplicity and utility to the scalar equation (2) of Sec. 4.4. The resultant polarized canonical form also suggests some interesting experimental programs that may be performed for polarized light fields in natural hydrosols. These will be briefly outlined at the conclusion of the section.

In order to establish the polarized version of (15) of Sec. 4.5, it seems natural to try to repeat the constructions between (1) and (15) of Sec. 4.5, now for each of the four components ${}_iN$ of the polarized observable radiance vector $\mathbf{N}$ (Sec. 2.10). Thus let us write:

$$\text{"}\mathbf{K}_i\text{"} \quad \text{for} \quad -\nabla_i N/{}_iN \tag{1}$$

for each component ${}_iN$ of $\mathbf{N}$, i = 1,2,3,4, and let us write (7) of Sec. 3.15 as:

$$\boldsymbol{\xi}\cdot\nabla\mathbf{N} = -\alpha\mathbf{N} + \mathbf{N}_* \tag{2}$$

where we have written:

$$\text{"}\mathbf{N}_*\text{"} \quad \text{for} \quad \int_\Xi \mathbf{N}\,\mathbf{p}\,d\Omega \tag{3}$$

where $\mathbf{p}$ is the standard observable volume scattering matrix. All that we need know about the standard observable volume scattering matrix $\mathbf{p}$ in the present derivation is that it is a 4 by 4 matrix with entry p_{ij} in the ith row and jth column.

In particular the problem of how the p_{ij} are obtained in principle or in practice is immaterial for the present derivation, since we are concerned only with the mathematical process of constructing the vector counterpart to (15) of Sec. 4.5. The matrix $\mathbf{p}$ is defined and discussed in detail in Sec. 112 of Ref. [251].

The canonical equation of transfer in the scalar context now becomes four coupled scalar equations in the polarized context as follows. We first write:

$$\text{"}\mathbf{p}_i\text{"} \quad \text{for} \quad (p_{1i}, p_{2i}, p_{3i}, p_{4i}) \tag{4}$$

Next we read off the ith component of (2), i = 1,2,3,4:

$$\xi \cdot \nabla {}_iN = - \alpha \, {}_iN + {}_iN_* \tag{5}$$

where we have written:

$$\text{"}{}_iN_*\text{"} \quad \text{for} \quad \int_{\Xi} \mathbf{N} \cdot \mathbf{p}_i \, d\Omega \tag{6}$$

It follows from (3) and (6) that:

$$\mathbf{N}_* = ({}_1N_*, {}_2N_*, {}_3N_*, {}_4N_*) \quad . \tag{7}$$

Using (1) in (5) and solving the result for ${}_iN$:

$$\boxed{{}_iN = \frac{{}_iN_*}{\alpha - \xi \cdot \mathbf{K}_i}} \tag{8}$$

This is the canonical equation of transfer for polarized radiance, which holds for each i = 1,2,3,4. Continuing as in Sec. 4.5, we deduce for i = 1,2,3,4:

$$ {}_iN_r / {}_iN_o = \exp \{ - \int_o^r \xi \cdot \mathbf{K}_i \, dr' \} \tag{9}$$

which is the vector component counterpart to (2) of Sec. 4.5. Applying the notation "$T_r[f]$" of Sec. 4.5 to the present context, (9) may be written:

$$ {}_iN_r = {}_iN_o T_r[-\xi \cdot \mathbf{K}_i] \quad , \tag{10}$$

and we observe that:

$$ {}_iN_r^o = {}_iN_o T_r[-\alpha] \quad . \tag{11}$$

It now follows readily that for every i = 1,2,3,4:

$$\boxed{{}_iN_r = {}_iN_oT_r[-\alpha] + \frac{{}_iN_*}{\alpha-\xi\cdot\mathbf{K}_i}\,(1-T_r[-(\alpha-\xi\cdot\mathbf{K}_i)])} \tag{12}$$

which is the desired *canonical representation of polarized radiance.*

The set of four equations (12) are coupled by means of the terms ${}_iN_*$. For example, the representation for ${}_1N_r$ uses ${}_1N_*$ where

$${}_1N_* = \int_\Xi \left[{}_1Np_{11} + {}_2Np_{21} + {}_3Np_{31} + {}_4Np_{41}\right] d\Omega$$

A Simple Model for Polarized Light Fields

We now give some attention to the construction of a simple model for polarized light fields in stratified natural hydrosols, the constructions being guided by the successful scalar prototype in Sec. 4.4. In the scalar case, the effective step was to assume that there was a nonnegative number K, less than α, such that:

$$N_*(z,\xi) = N_*(z_o,\xi)e^{-K(z-z_o)} \quad . \tag{13}$$

This suggests that we take each ${}_iN_*$, i = 1,2,3,4, which by (6) has the form:

$${}_iN_* = \int_\Xi \left\{{}_1Np_{1i} + {}_2Np_{2i}\ {}_3Np_{3i} + {}_4Np_{4i}\right\} d\Omega \tag{14}$$

and agree to write:

$$\text{"}{}_{ji}N_*\text{"} \quad \text{for} \quad \int_\Xi {}_jNp_{ji}\ d\Omega \quad , \tag{15}$$

so that ${}_iN_*$ will have the representation:

$${}_iN_* = {}_{1i}N_* + {}_{2i}N_* + {}_{3i}N_* + {}_{4i}N_* \quad . \tag{16}$$

Then, still being guided by the prototype (13) we agree to make the following assumption: the four nonnegative real numbers K_i, as defined in (1), are each less than α, and such that:

$$\boxed{{}_{ji}N_*(z,\xi) = {}_{ji}N_*(z_o,\xi)e^{-K_j(z-z_o)}} \tag{17}$$

for every i, j = 1,2,3,4, where K_i now is the z-component of $\mathbf{K}_j$--the only nonzero component of $\mathbf{K}_j$ by virtue of our current assumption about the stratification of the light field in natural waters. Under these assumptions, (12) reduces to:

$$\boxed{{}_iN_r(z,\xi) = {}_iN_o(z_o,\xi)e^{-\alpha r} + \frac{{}_iN_*(z,\xi)}{\alpha + K_i \cos\theta}\left[1 - e^{-(\alpha + K_i \cos\theta)r}\right]} \tag{18}$$

for i = 1,2,3,4, and where:

$$\begin{aligned} {}_iN_*(z,\xi) &= {}_{1i}N_*(z_o,\xi)e^{-K_1(z-z_o)} \\ &+ {}_{2i}N_*(z_o,\xi)e^{-K_2(z-z_o)} \\ &+ {}_{3i}N_*(z_o,\xi)e^{-K_3(z-z_o)} \\ &+ {}_{4i}N_*(z_o,\xi)e^{-K_4(z-z_o)} \end{aligned}$$

or more compactly:

$$\boxed{{}_iN_*(z,\xi) = \sum_{j=1}^{4} {}_{ji}N_*(z_o,\xi)e^{-K_j(z-z_o)}} \tag{19}$$

Experimental Questions

The derivation of the canonical representation (18) for polarized radiance incorporated several assumptions which, even though suggested by the well-established scalar case of Sec. 4.4, require some critical examination before they are fully accepted. These assumptions in turn raise certain specific questions concerning the nature of polarized light fields in natural hydrosols in general and the nature of the K-functions in particular. We shall conclude the present section with a brief statement and discussion of these questions.

First of all, the definition of each K_i as given in (1), is a constructive definition and hence presents no difficulty in being translated into operational terms, so that actual experimental determinations of the K_i are possible in principle. These determinations should parallel very closely those already developed for the function K in (20) of Sec. 4.5, because K_i, as K, is a logarithmic derivative of a radiance function. The main difference between K_i and K is simply that each K_i is associated with the component of a vector valued function while K is associated with a scalar valued function. Thus with the extra attachments of wave plates and polarizers on the radiance meter required to measure the polarized radiance, one performs essentially those

operations with the radiance meter that one performs to find K, but now four times over for each K_i, i = 1,2,3,4.

With the matter of the measurability of the K_i settled, at least in principle, we now ask the first question that comes to mind concerning the K_i: Is there some observable regular pattern in the individual depth-behavior and in the relative magnitudes of the four functions K_1, K_2, K_3, K_4? This is actually two questions in one, and it may be simpler to phrase them separately. The first question may be phrased: *Is there some observable regularity in the depth behavior of each K_i?* The second question may then be rendered as: *Is there some observable regularity in the relative magnitudes of the K_i?* As far as the first question is concerned, it is expected on simple physical grounds that the individual depth behavior of each K_i should follow very closely that of the scalar K defined in (20) of Sec. 4.5. In particular the depth behavior of the K_i at relatively great depths in homogeneous media should be quite regular and should follow the patterns discussed in Sec. 7.10 and Sec. 10.6 dealing with the asymptotic radiance theorem. Some attention to this question has been given by Lenoble [157]. The second question is more difficult to answer and, in view of the present state of development of the theory of polarized light fields in natural optical media, it appears likely that a definitive answer will be forthcoming first from experimental investigations. Nevertheless, it is interesting to speculate on the possible interrelations among the K_i. Thus, suppose that the K_i are all equal to a common value, then the set of equations in (18) assumes a particularly simple form. It follows that any differences between ${}_iN_r$ and ${}_jN_r$ will depend solely on the state of affairs between them at the surface of the medium. On the other hand, if there are two K_i's which differ at all depths then the radiance component associated with the larger K_i will decay with depth more quickly than the other. As a result, those components of N with the smallest K_i's will persist down to greater depths than the others with larger K_i's. By contemplating these possibilities and by taking into account the known properties of the unpolarized light field, the general state of affairs for the functions K_i will most likely turn out as follows: Near the surface the K_i's will differ, and there will be some permanent characteristic pattern of relative sizes discernible among them which is related to the state of the sea surface, and to the polarized state of the sky; however, the transmitted sky-polarization and under-surface reflection-induced pattern will eventually disappear with increasing depth in such a manner that in the limit, all the values K_i tend to a common value independent of the state of the sky's polarization, with an attendant asymptotic value of the polarization of the light field. This common value of the K_i's will be that of the depth decay rate of scalar irradiance h(z), which should be determined only by the inherent optical properties of the medium, just as in the scalar case. It remains to be seen how this conjecture is borne out by experimental studies. Our review of the experimental work of Ivanoff and Waterman in 1.2 shows some encouraging agreement in this direction.

While attention is directed toward the possible structure of the functions K_i at great depths and while conjectures

about the K_i are being made, it might be in place to add some further conjectures about the light field itself in addition to its depth-rates of decay K_i. When one imagines the natural light field at great depths one is led to picture a predominantly downward feeble flow of light, the radiance pattern being graphically depicted by an ellipsoid-like surface with vertical axis. If this light field is conceptually analyzed for polarization features, it seems--on intuitive grounds--that the radiance vector for vertical downward or upward flux should have the form (1/2)(N,N,N,N), i.e., vertical downward or upward-radiance should be unpolarized. Furthermore, it seems that the horizontal radiance should be horizontally linearly polarized, i.e., have the form (1/2) × (0,2N,N,N). This follows from the fact that the flow is predominantly vertical and beamlike (and of course very feeble) at great depths. Since natural light fields change continuously rather than abruptly in most macroscopic settings, we would expect the radiance vector components to vary continuously between these two extremes as the angle of the radiance direction varies from $\theta = 0$ (vertical upward), or π (vertical downward) to $\theta = \pi/2$ (horizontal). A simple model for this radiance $N(\theta)$ which comes readily to mind and which satisfies these conditions is:

$$N(\theta) = \frac{1}{2}\,(N\cos^2\theta,\ N(1+\sin^2\theta),\ N,\ N)$$

where θ is measured from the zenith and N is the fixed references radiance for $\sigma = o$ at each depth. All these assertions are at this stage of our knowledge of course conjectural, being based on a modicum of physical experience with polarized radiance fields in natural waters, and are intended primarily to perform a heuristic service. It will be left to interested researchers to carry this matter to a more satisfactory state of affairs, both theoretically and experimentally. A possible theoretical approach can be based on the polarized version of (21) of Sec. 10.7, or on (29), (31) of Sec. 7.10. These approaches may show that the preceding conjecture must be modified to take into account the structure of the volume scattering matrix (cf. (24) of Sec. 13.6) of the medium.

4.7 Abstract Versions of Canonical Equations

The discussions of the present chapter have carried the notion of canonical radiance forms over a great conceptual distance, starting from the rudimentary canonical representation (1) of Sec. 4.0 discovered by Bouguer nearly two centuries ago and up to the representation in (12) of Sec. 4.6. Such a task could not have been carried out in the indicated manner without the convenient milestones in the development of the theory provided by early workers such as Schuster, Koschmieder, and others. It seems that the representations finally reached in Secs. 4.5 and 4.6 constitute the most general forms for radiance concepts attainable which are physically meaningful. Their basic forms remain essentially intact by allowing more general physical features to appear such as the time-dependent radiance terms

and emission terms in the basic equation of transfer. In view of the apparent ubiquity of the canonical representation throughout the domain of pure and applied radiative transfer theory (e.g., see the canonical equations in Chapter 11) and in view of the seeming ease with which the equation of transfer is molded into its canonical form, we are led to inquire whether the notion of a canonical representation is indigenous only to radiative transfer theory or whether in our labors in this special field we have touched upon merely the shadow or projection, so to speak, of a more general analytic phenomenon in modern operator theory. It appears that the latter possibility is the case and we pause briefly here to sketch in outline the general mathematical setting in which the notion of the canonical representation appears to take a natural place.

Let L be a general (not necessarily linear) operator defined on a domain $\mathfrak{D}$ of functions such that for each function f in $\mathfrak{D}$ there is a function g in $\mathfrak{D}$ and a number λ such that:

$$\boxed{Lf = \lambda f + g} \tag{1}$$

This is the abstract counterpart to the equation of transfer with L replacing the derivative operation $\xi \cdot \nabla$, and g replacing N_*, and where f replaces N. The number λ is nonzero and may be real or complex and is evidently a replacement of $-\alpha$. Now let us write:

$$\text{"}f_q\text{"} \quad \text{for} \quad -g/\lambda \quad .$$

Then (1) can be written:

$$\boxed{Lf = \lambda(f-f_q)} \tag{2}$$

and this should be compared with (4) of Sec. 4.3. Hence f_q is the abstract vestige of equilibrium radiance, so that $Lf = 0$ if and only if $f = f_q$. Next write

$$\text{"}\kappa\text{"} \quad \text{for} \quad -Lf/f \tag{3}$$

so that κ is the abstract vestige of $\mathbf{K}$, and (1) becomes:

$$-\kappa f = \lambda f + g \quad .$$

Solving this for f:

$$\boxed{f = \frac{-g}{(\lambda + \kappa)}} \tag{4}$$

which is the requisite abstract canonical form of equation (1) associated with the operator L. An alternate form of (4) is obtained by using f_q:

$$\boxed{f = \frac{f_q}{1 + (\kappa/\alpha)}} \quad . \tag{5}$$

This basic form is applicable to all manners of radiometric concepts and optical properties. See, e.g., the various specific forms of (5) appearing throughout Chapter 11.

The abstract version of the canonical representation of f now follows readily from (4) or (5), by emulating (10) of Sec. 4.5. Now that a decomposition of f into "reduced" and "diffuse" may not be natural, we simply represent f by the identity:

$$f = fT + f(1-T) \quad , \tag{6}$$

where T is any suitable operator on $\mathfrak{D}$ and "1" denotes the identity transformation on $\mathfrak{D}$. Then using (4), this becomes:

$$\boxed{f = fT + \frac{g}{(\lambda + \kappa)}(T-1)} \quad , \tag{7}$$

which is an abstract canonical representation of f with respect to the operators T and L, via equation (1), and is to be compared to (15) of Sec. 4.5.

A more direct generalization of (15) of Sec. 4.5 (which retains the idea of "diffuse" and "reduced" components) follows upon replacing (6) by:

$$f = f_o + (f - f_o) \tag{8}$$

and defining two operators S and T such that there exists a function ϕ_o with the property that

$$f_o = \phi_o\, T \qquad \text{(cf. (12) of 4.5)} \tag{9}$$

$$f = \phi_o\, S \qquad \text{(cf. (13) of 4.5)}$$

With these definitions (8) becomes

$$f = \phi_o\, T + (\phi_o S - \phi_o T)$$

whence

$$f = \phi_o T + f(1-S^{-1}\, T) \quad . \tag{10}$$

Let us write

$$\text{"}\tau\text{"} \quad \text{for} \quad S^{-1}\, T$$

then we obtain,

$$f = \phi_0 T + f(1-\tau) \tag{11}$$

which with (4) becomes:

$$\boxed{f = \phi_0 T + \frac{-g}{(\lambda + \kappa)} (1-\tau)} \quad . \tag{12}$$

This is the requisite abstract version of (15) of Sec. 4.5, and the ultimate generalization of (1) of Sec. 4.0 to be attempted here. We say that (12) is the *canonical representation of f with respect to the operators L, T, S, via the equation (1).* The operator τ turns out to be the abstract counterpart to the contrast transmittance function (Sec. 9.5).

By performing the preceding constructions of the abstract version of the canonical representation we gain a deep insight into the essential mathematical structure of the canonical representations in radiative transfer theory. Our constructions show us, in particular, that the essential physical kernel of (12) is bound up in the term $-g/(\lambda + \kappa)$, and that the overall general structure of (12), as given by (8) or (11), is a mere mathematical tautology. It seems somewhat noteworthy, therefore, that Bouguer, who discovered the first definitive trace of the canonical equation in the form (1) of Sec. 4.0, managed to light upon the essential form but yet with only partial realization of the significance of the two key physical terms a and b of the canonical form. The lessons of this chapter and hindsight now let us see that within the apparently insignificant term b, as it occurs in (1) of Sec. 4.0, resides not only the notion of equilibrium radiance, but actually the equation of transfer for radiance, the basic law of all of radiative transfer theory.

4.8 Bibliographic Notes for Chapter 4

One of the earliest known instances of the canonical form of the equation of transfer was written down by Bouguer in his classical treatise on light, recently translated by Middleton [28]. The equation appears in essentially the form it is closest to the basic integral representation of the equation of transfer as given in (5) of Sec. 3.13. Soon after Schuster formulated his celebrated two-flow equations [279], Schwarzschild [281] in 1906 formulated an expression for what we now call "path radiance", and later, in 1914, Schwarzschild [282] incorporated it into an expression for radiance, which is essentially (6) of Sec. 3.13. The latter equation was our point of departure from which we deduced the classical form of the canonical equation, as given in (2) of Sec. 4.4.

It appears from a perusal of the literature that the canonical form of the equation of transfer, as embodied, say in (2) of Sec. 4.4, took its first definitive general form in [212] and [250] which in turn grew out of the hydrologic optics researches recorded in [82] and [5]. However, as noted in the introductory remarks, the canonical form in one

guise or other from (1) of Sec. 4.0 to (11) of Sec. 4.4 (and even (8) of Sec. 5.3) appears and reappears in the work of independent researchers over the years in diverse applications of radiative transfer. One outstanding early use of the canonical equation is in the work of Koschmieder [141]. The task of tracing the subsequent manifold reappearances of the canonical form is best left to historians of the subject. One source of references for such work is Middleton's treatise [177]. For our purposes it suffices to anchor the canonical equation's first ground-form in Schwarzschild's work [282], as noted above. It has been the intention of this chapter to clarify the canonical equation's logical and conceptual roles in the general theory of radiative transfer as outlined in Secs. 4.4, 4.5, and 4.7, and to extend it to the polarized context as in Sec. 4.6. For further discussions of underwater polarized light fields, see [117], [118], [108].

CHAPTER 5

NATURAL SOLUTIONS OF THE EQUATION OF TRANSFER

5.0 Introduction

The natural solution of the equation of transfer plays a fundamental and unique role in the theory of radiative transfer. The role is fundamental in the sense that the natural solution may be used in the systematic construction of the principles of invariance, the invariant imbedding principle, and all other instances of the interaction principle encountered in radiative transfer theory. This facet of the natural solution was explored in an earlier study [251] and so need not be considered in detail in the present work. The uniqueness of the role of the natural solution of the equation of transfer lies in its remarkably wide-ranged interpretation. On the one hand, the natural solution affords a simple intuitive picture of multiply scattered light in natural media; on the other hand it forms a link with certain general iterative solution procedures of functional equations in modern operator mathematics. No other extant mode of solution of the equation of transfer possesses such a combination of intuitive and formal features. In the present chapter we shall concentrate on these features of the natural solution, with particular emphasis on the intuitive insight into the concept of multiply scattered light in optical media supplied by the natural solution.

We shall first consider the intuitive features of the natural solution. These features will be of help to the reader in the task of following all the formal developments of the present chapter and will also help build a working intuition about natural light fields in general. We begin by observing that the natural solution of the equation of transfer is based on the idea of the scattering order decomposition of a light field. This idea in turn is based on the premise that radiant flux pouring into a medium past its boundaries generates multiply scattered radiant flux within the medium and that this radiant flux is subject to a precise mathematical analysis. It is the task of the natural mode of solution of the equation of transfer to first of all unravel the apparently chaotic resultant jumble of radiant flux of all scattering orders and arrange the flux in an orderly, countably infinite sequence of indexed flows, i.e., of integer-numbered scattering orders, and then to relate each of the indexed flows by means of well-defined formulas to the other indexed flows representing the higher and lower

scattering orders. These features of the natural solution can be seen in detail with the help of a simple analogy which we shall now consider.

Consider a lake on a clear sunny day. Sunlight and skylight stream down and enter the lake surface, penetrate into the body of the medium, are partially absorbed and partially scattered throughout the body of the lake, and eventually the scattered light comes to a general steady state of flow in the various directions about each point of the medium. Now we may liken the incident radiant flux on the lake surface to a family of tiny colored particles (the geometric vestige of photons of given frequency), and we may liken the substance of the lake, in reality an aggregate of molecules of water, minerals, and organic materials, to a set X of stationary bodies distributed in space, and relatively massive with respect to the incident particles. The interaction of the photons with the lake molecules may then be envisioned, for the purposes of the present discussion, in terms of the interactions of tiny colored particles with the members of an aggregate of relatively massive stationary bodies. Then within this setting, the caroming of a tiny colored particle off the side of a massive body without change in color of the particle may be interpreted as a scattering operation not unlike the elastic scattering of a photon by a molecule; and the permanent absorption of a particle of given color by a body may be thought of as the analog of an absorption by a molecular field of a photonic field's energy.

Within the present simplified setting consisting of a swarm of colored particles migrating through a maze of relatively massive stationary bodies, the natural mode of solution of the equation of transfer takes the following form. The natural mode of solution partitions the complex steady state flow of an arbitrary given set P of monochromatic particles through the space X into sets of separate families P_n of particles. Each family P_n of particles is a subfamily of P and is identified by its scattering order, n, that is by an integer n representing the common number of scatterings undergone by each particle in the family since the particle entered the medium X. Thus at some arbitrary fixed instant t in time let $P_0(t)$ be the family of particles throughout X which have not undergone any scattering in X subsequent to entering X. In general, let $P_n(t)$ be the family of swarming particles throughout X which have undergone precisely n scatterings in X, since the particles entered X, where n is a nonnegative integer. Hence at each instant t we conceptually partition the collection P(t) of colored particles within X into an ordered, pairwise disjoint collection $P_0(t)$, $P_1(t)$, ..., $P_n(t)$, ..., of particles. This ordered collection is called the *scattering order decomposition* of the light field. Whenever a member of $P_n(t)$ undergoes a scattering event at time $t + \Delta t$ where $\Delta t \geq 0$, it enters the family $P_{n+1}(t + \Delta t)$. In the steady state, the number of members of $P_n(t)$ is independent of t.

Now the members of $P_n(t)$ are generally to be found flowing in every direction within the neighborhood of any point within X. This flow in the neighborhood of the point has assignable, at least on the conceptual level, a radiance

$N^n(t)$. The natural representation of the radiance field in this setting is then defined as the sum $\sum_{n=0}^{\infty} N^n(t)$ of the radiances associated with all the $P_n(t)$. A radiance function obtained in this manner in an optical medium will be shown to be a solution--the natural solution--of the equation of transfer for that optical medium.

5.1 The n-ary Radiometric Concepts

In this section we shall define those radiometric concepts associated with the scattering order decomposition of a light field which will be needed in the developments of the present chapter. Throughout this section we work with a general source-free optical medium X in the steady state irradiated by a steady incident radiance function N_o defined on the boundary of X. The medium X is generally inhomogeneous, of arbitrary shape and extent, and with general volume attenuation and scattering functions defined throughout. The incident radiance associated with N_o penetrates into X and generates radiant flux of arbitrarily great scattering orders, which we now proceed to analyze.

n-ary Radiance

The systematic construction of the radiance functions associated with the families $P_n(t)$ of photons described in the introductory section starts with the incident radiance N_o on the boundary of X. In particular, the radiance $N_o(x_o,\xi)$ defined for a boundary point x_o and the direction ξ at x_o can be extended to each point x of X by writing:

$$"N^o(x,\xi)" \quad \text{for} \quad N_o(x_o,\xi)T_r(x_o,\xi) \tag{1}$$

where $x = x_o + r\xi$. The meanings of these terms are shown in Fig. 5.1. In this way we can construct a radiance distribution $N^o(x,\cdot)$ at each point x inside and on the boundary of X. We call N^o the *initial (residual or unscattered or reduced) radiance function* within X. N^o represents radiance which, relative to the radiance N_o incident on the boundary of X, has undergoine no scattering operations within X.

When some of the flux which comprises the initial radiance distribution $N^o(x\cdot)$ at x undergoes a scattering operation there is generated first order (or primary) scattered radiant flux. The amount generated per unit length in the direction ξ at x is represented by writing:

$$"N_*^1(x,\xi)" \quad \text{for} \quad \int_{\Xi} N^o(x,\xi')\sigma(x;\xi';\xi)\, d\Omega(\xi') \tag{2}$$

This may be written succinctly in operator form using the path function operator $\mathbf{R}$ of Sec. 3.17:

$$N_*^1 = N^o\mathbf{R} \quad . \tag{3}$$

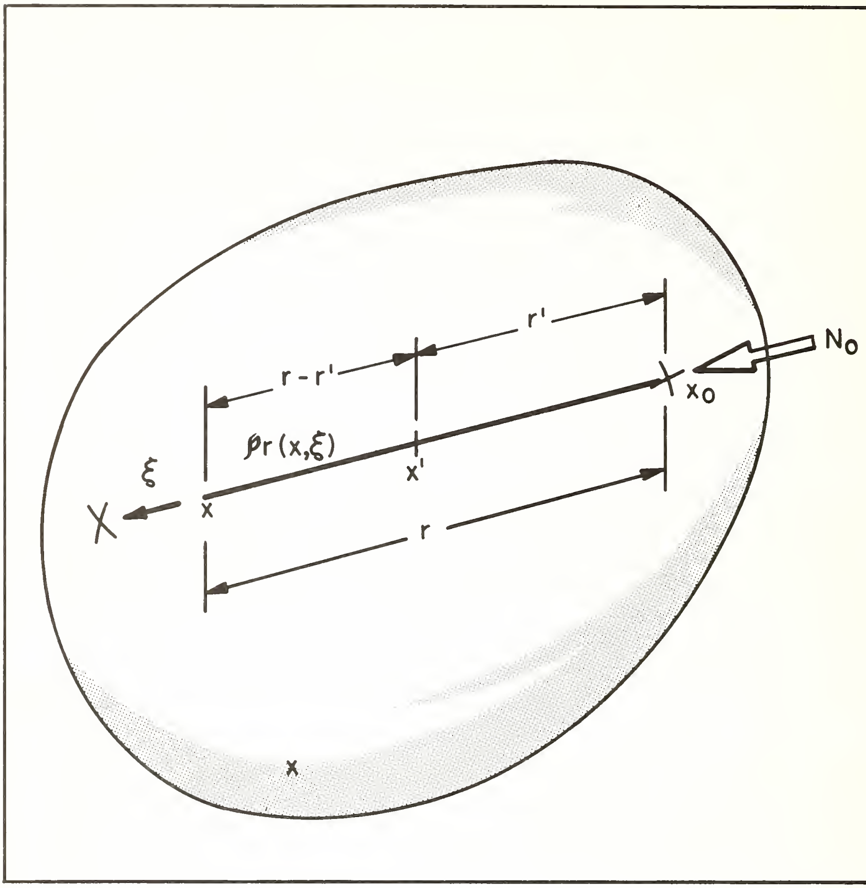

FIG. 5.1 Geometric details for computing n-ary radiance.

In other words, the operator **R** acts on N^0 to generate N^1_*; alternately, we may say that **R** maps N^0 into N^1_*. The amount of primary scattered radiance accumulated over a path $\mathcal{P}_r(x,\xi)$ in X is then represented by writing:

$$\text{"}N^1(x,\xi)\text{"} \quad \text{for} \quad \int_0^r N^1_*(x',\xi)\, T_{r-r'}(x',\xi)\, dr' \quad . \tag{4}$$

This may also be written succinctly using the path radiance operator $\mathbf{T}$ of Sec. 3.17:

$$N^1 = N^1_* \mathbf{T} \quad . \tag{5}$$

The general pattern of construction of the radiance functions comprising the scattering order decomposition of the light field should now be clear. Thus, for every integer $n = 0,1,2, \ldots$, we agree to write:

$$\text{"}N_*^{n+1}\text{"} \quad \text{for } N^n\mathbf{R} \tag{6}$$

and

$$\text{"}N^{n+1}\text{"} \quad \text{for } N_*^{n+1}\mathbf{T} \tag{7}$$

The function N_*^n is called the *n-ary path function* and N^n is the *n-ary radiance function* in X relative to N^o. By means of (6) and (7) we can construct the (n+1)-ary radiance function on X once we know the n-ary radiance function on X, for $n \geq 0$ thus:

$$\begin{aligned} N^{n+1} &= N_*^{n+1}\mathbf{T} = (N^n\mathbf{R})\mathbf{T} \\ &= N^n(\mathbf{RT}) \end{aligned} \tag{8}$$

for every scattering order $n \geq 0$. The composition **RT** of the two operators **R** and **T** occurs often in our studies of radiative transfer theory. We shall then write, for brevity:

$$\text{"}\mathbf{S}^1\text{"} \quad \text{for } \mathbf{RT} \tag{9}$$

The reader should verify that:

$$\boxed{\mathbf{S}^1 = \int_0^r \left[\int_\Xi \left[\ \right] \sigma(x';\xi';\xi)\, d\Omega(\xi') \right] T_{r-r'}(x',\xi)\, dr'} \tag{10}$$

Now, using the notation for $\mathbf{S}^1$, (8) may be written:

$$\boxed{N^{n+1} = N^n\mathbf{S}^1} \quad , \tag{11}$$

and if n is an arbitrary integer greater than 0, then it follows that we can apply the statement (8), or statement (11), once again to obtain:

$$N^{n+1} = (N^{n-1}\mathbf{S}^1)\mathbf{S}^1 \tag{12}$$

If $n-1 \geq 1$, then we can apply (11) again, with the eventual conclusion that N^{n+1} is represented as the result of operating on N^o with $\mathbf{S}^1$ at total of n+1 times in succession. That is, if we write:

$$\text{"}\mathbf{S}^{n+1}\text{"} \quad \text{for } \mathbf{S}^1\mathbf{S}^n \tag{13}$$

for every integer n, $n \geq 0$, then it is an easy application of the principle of complete induction to show that:

$$\boxed{N^n = N^o \mathbf{S}^n} \tag{14}$$

for every scattering order (nonnegative integer) n. The sense in which (13) and (14) are to be understood is the obvious one: Operate on N^o and $\mathbf{S}^1$ to obtain N^1; then once N^1 is obtained, operate on N^1 with $\mathbf{S}^1$ to obtain N^2; and so on until N^n is obtained. The total combined integration operation of obtaining n-ary radiance N^n from the initial radiance N^o is summarized by the operator $\mathbf{S}^n$ defined recursively in (13).

n-ary Scalar Irradiance

Now that the n-ary radiance functions have been defined it is a relatively easy matter to define the n-ary counterparts to all the radiometric concepts. For example, by recalling the integral representation of scalar irradiance h(x) at a point x in the optical medium X (cf. Sec. 2.7), i.e., the definition in which we have written:

$$\text{"}h(x)\text{"} \quad \text{for} \quad \int_{\Xi} N(x,\xi)\,d\Omega(\xi) \quad ,$$

we are then led to write analogously:

$$\text{"}h^n(x)\text{"} \quad \text{for} \quad \int_{\Xi} N^n(x,\xi)\,d\Omega(\xi) \tag{15}$$

for every nonnegative integer n. We call $h^n(x)$ the *n-ary scalar irradiance* in X relative to N^o.

n-ary Radiant Energy

The connection between scalar irradiance h(x) and radiant density u(x) at each point x of X was seen in Sec. 2.7 to be:

$$h(x) = v(x)u(x)$$

where v(x) is the speed of light at x in X. Furthermore the definition of the radiant energy content U(X) of X was defined by writing:

$$\text{"}U(X)\text{"} \quad \text{for} \quad \int_{X} u(x)\,dV(x) \quad .$$

This leads us to write analogously:

$$"U^n(x)" \quad \text{for} \quad \int_X u^n(x)\,dV(x) \tag{16}$$

for every nonnegative integer n where, in turn, we have written:

$$"u^n(x)" \quad \text{for} \quad h^n(x)/v(x) \tag{17}$$

for every nonnegative integer n. Combining the definitions of h^n, u^n and U^n, we have the following representation of U^n:

$$U^n(X) = \int_X \frac{1}{v(x)} \left[\int_\Xi N^n(x,\xi)\,d\Omega(\xi) \right] dV(x) \tag{18}$$

for every nonnegative integer n, and where the n-ary radiance N^n is represented in terms of the initial radiance N^o throughout X by means of (14).

General n-ary Radiometric Functions

The n-ary radiance and radiant energy functions constructed above will not be the only n-ary radiometric concepts used in the present work. For example the two-flow equations of Sec. 8.4 are studied by means of n-ary irradiance concepts. It is a simple matter to extend the type of definition exhibited for h^n and U^n to an arbitrary function C obtained from the radiance function by an appropriate linear operator $\mathcal{L}$ associated with C; that is:

$$C = N\mathcal{L} \quad . \tag{19}$$

For example, the operator $\mathcal{L}$ in the case where C is scalar irradiance was:

$$\mathcal{L} = \int_\Xi [\ \ ]\,d\Omega(\xi) \quad .$$

Then in general we write analogously:

$$"C^n" \quad \text{for} \quad N^n\mathcal{L} \quad , \tag{20}$$

for every nonnegative integer n. We call C^n the *n-ary radiometric function* of C, in X, and relative to N^o. It follows from (14) and (2) that:

$$C^n = N^o(S^n\mathcal{L}) \tag{21}$$

is the representation of the n-ary radiometric function C^n associated with the general radiometric concept C. In particular, we write:

$$\text{"}C^*\text{"} \quad \text{for} \quad N^* \mathscr{L} \tag{22}$$

where N* is the path radiance (the scattered) component of N, as it occurs in (5) of Sec. 3.13. C* is the *diffuse radiometric function of C* in X and relative to N^o. Together, C* and C^n are the *decomposed* radiometric functions. Radiometric functions which have not been decomposed are call *undecomposed.*

5.2 Equation of Transfer for n-ary Radiance, Diffuse Radiance, and Path Function

The equation of transfer for n-ary radiance will now be derived. The equation is an elementary consequence of relation (11) of Sec. 5.1. To see this, suppose we fix attention on an arbitrary path $\mathscr{P}_r(x,\xi)$. Then holding the initial point x and the direction ξ of the path fixed, and differentiating N^n along the path with respect to path length r, we have:

$$\frac{dN^n}{dr} = \frac{d}{dr}\left(N^{n-1}s^1\right)$$

$$= \frac{d}{dr}\int_0^r \left[\int_\Xi N^{n-1}(x',\xi')\sigma(x';\xi';\xi)\,d\Omega(\xi')\right] T_{r-r'}(x',\xi)\,dr'$$

$$= \int_\Xi N^{n-1}(x,\xi')\sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$+ \int_0^r \left[\int_\Xi N^{n-1}(x',\xi')\sigma(x';\xi';\xi)\,d\Omega(\xi')\right] \frac{dT_{r-r'}(x',\xi)\,dr'}{dr}$$

At this point we observe that, by (3) of Sec. 3.11:

$$\frac{dT_{r-r'}(x',\xi)}{dr} = -\alpha(x,\xi)T_{r-r'}(x',\xi)$$

Then using (6) and (11) of Sec. 5.1 we arrive at:

$$\boxed{\xi\cdot\nabla N^n = \frac{dN^n}{dr} = -\alpha N^n + N^n_*} \tag{1}$$

which is the requisite *equation of transfer for n-ary radiance* with $n \geq 1$. Observe that the equation of transfer for N^n is not an integrodifferential equation for N^n; rather it

is a first order linear differential equation for N^n with known n-ary path function N_*^n, once N^{n-1} is known. This suggests a conceptually powerful natural mode of solution of the general equation of transfer for N, which we shall study throughout this chapter. In the following section we shall place (1) into its canonical form, thus rounding out the studies of the canonical equation given in Chapter 4. In Sec. 5.4, the complete natural solution will be obtained.

Before concluding this discussion on n-ary radiance equations, we mention two more transfer equations for radiometric concepts which are closely related to the family of equations in (1). Note that (1) holds only for $n \geq 1$, the case $n = 0$ being excluded. This singular case $n = 0$ is readily stated using (4) of Sec. 3.10 and (2) of Sec. 3.11. The result is:

$$\boxed{\xi \cdot \nabla N^o = \frac{dN^o}{dr} = -\alpha N^o} \tag{2}$$

for source-free media. A generalization of (2) for media with sources is given in (2) of Sec. 5.8. The remaining transfer equation to be noted here is that for the *diffuse radiance* N* (or path radiance when a specific path of length r is given somewhere in the medium). Thus, using the concept of n-ary radiance, let us write:

$$\text{"N"} \quad \text{for} \quad \sum_{j=0}^{\infty} N^j \tag{3}$$

$$\text{"N*"} \quad \text{for} \quad \sum_{j=1}^{\infty} N^j \tag{4}$$

and

$$\text{"}N_*^*\text{"} \quad \text{for} \quad \sum_{j=2}^{\infty} N_*^j \quad . \tag{5}$$

Then summing each side of (1) over all n from 1 to ∞, we have:

$$\sum_{j=1}^{\infty} \xi \cdot \nabla N^j = \sum_{j=1}^{\infty} \frac{dN^j}{dr} = -\sum_{j=1}^{\infty} \alpha N^j + \sum_{j=1}^{\infty} N_*^j \tag{6}$$

which, on applying (4) and (5) becomes:

$$\boxed{\xi \cdot \nabla N^* = \frac{dN^*}{dr} = -\alpha N^* + N_*^* + N_*^1} \tag{7}$$

This is the equation of transfer for diffuse radiance N*. By assuming that N_*^* obeys (1) of Sec. 4.4, i.e., N_*^* decays exponentially with depth at the rate K, then (7) supplies a somewhat more powerful description of the light field than

that given by (2) of Sec. 4.4. It is clear from the discussions of Sec. 5.1 and (5) that:

$$N_*^* = N^*\mathbf{R} \tag{8}$$

We shall return to these ideas in Sec. 5.4, especially to that of N, as defined in (3), wherein we will show that N so defined is a solution of the equation of transfer.

Finally, by applying the operator $\mathbf{R}$ to each side of the equation of transfer for radiance, we find:

$$\boxed{\xi \cdot \nabla N_* = -\alpha N_* + N_{**}} \tag{9}$$

which is the equation of transfer for the path function, and where we have written:

$$\text{"}N_{**}\text{"} \quad \text{for} \quad N_*\mathbf{R} \tag{10}$$

5.3 Canonical Equations for n-ary Radiance

We pause in the present development of the natural solution of the equation of transfer to present the canonical form of the transfer equation for n-ary radiance. We shall be particularly interested in the case of n = 1, that is, in the case of the canonical equation for primary radiance. From this case we can derive an expression which has often formed an integral part of expressions which attempt to approximately represent radiance distribution with a modicum of analytic complications. The derivations below are patterned on those in Sec. 4.5. Hence we can proceed with a minimum of motivation and explanation for the present discussion. Let us write:

$$\text{"}\mathbf{K}^n\text{"} \quad \text{for} \quad -\frac{\nabla N^n}{N^n} \tag{1}$$

Then (1) of Sec. 5.2 becomes:

$$-\xi \cdot \mathbf{K}^n N^n = -\alpha N^n + N_*^n \quad ,$$

whence, for every integer n with $n \geq 1$:

$$\boxed{N^n = \frac{N_*^n}{\alpha - \xi \cdot \mathbf{K}^n}} \tag{2}$$

and consequently:

$$\boxed{N_r^n = N_o^n T_r[-\alpha] + \frac{N_*^n}{\alpha - \xi \cdot \mathbf{K}^n}\left(1 - T_r[-(\alpha - \xi \cdot \mathbf{K}^n)]\right)} \tag{3}$$

which are respectively, the desired *canonical form of the equation of transfer* for n-ary radiance and its *canonical representation for a path* $\mathcal{P}_r(x,\xi)$.

If the medium X is assumed to be a plane-parallel stratified optical medium, then following the pattern established in equations (16) - (19) of Sec. 4.5, (2) and (3) reduce to:

$$N^n = \frac{N_*^n}{\alpha + K^n \cos\theta} \tag{4}$$

and the associated canonical representation of N_r^n over a path $\mathcal{P}_r(x,\xi)$, analogous to N_r of Sec. 4.5 is:

$$N_r^n = N_o^n T_r[-\alpha] + \frac{N_*^n}{\alpha + K^n \cos\theta}\left\{1 - T_r[-(\alpha + K^n \cos\theta)]\right\} \tag{5}$$

Equations (2) and (3), and their special cases (4) and (5), are the alternate (canonical) ways of representing N^n; the usual way being summarized in (14) of Sec. 5.1 by:

$$N^n = N^o \mathbf{S}^n \tag{6}$$

To see how (2), (3), and (6) throw light on one another, let us consider the case of a homogeneous source-free plane parallel medium X irradiated by narrow beams of radiance N_o incident at each point of its upper boundary through a small solid angle Ξ_o of magnitude Ω_o, as shown in Fig. 5.2. The radiant flux from N_o initiates a multiple scattering process within X and eventually all scattering orders of radiant flux are present within X. We direct attention now to N^1 and first compute its value at depth z in the direction ξ using (6). Thus, from (6) with n = 1:

$$N^1 = (N^o \mathbf{R})\mathbf{T} \quad .$$

For the present case $N^o\mathbf{R}$ is readily evaluated:

$$N_*^1(z,\xi) = N^o\mathbf{R} = \int_{\Xi} N^o(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad .$$

Since for each ξ' in Ξ_o,

$$\begin{aligned} N^o(z,\xi') &= N_o(0,\xi')T_r(0,\xi') \\ &= N_o e^{\alpha z \sec\theta_o} \end{aligned}$$

where:

$$\cos\theta_o = \xi_o \cdot \mathbf{k}$$

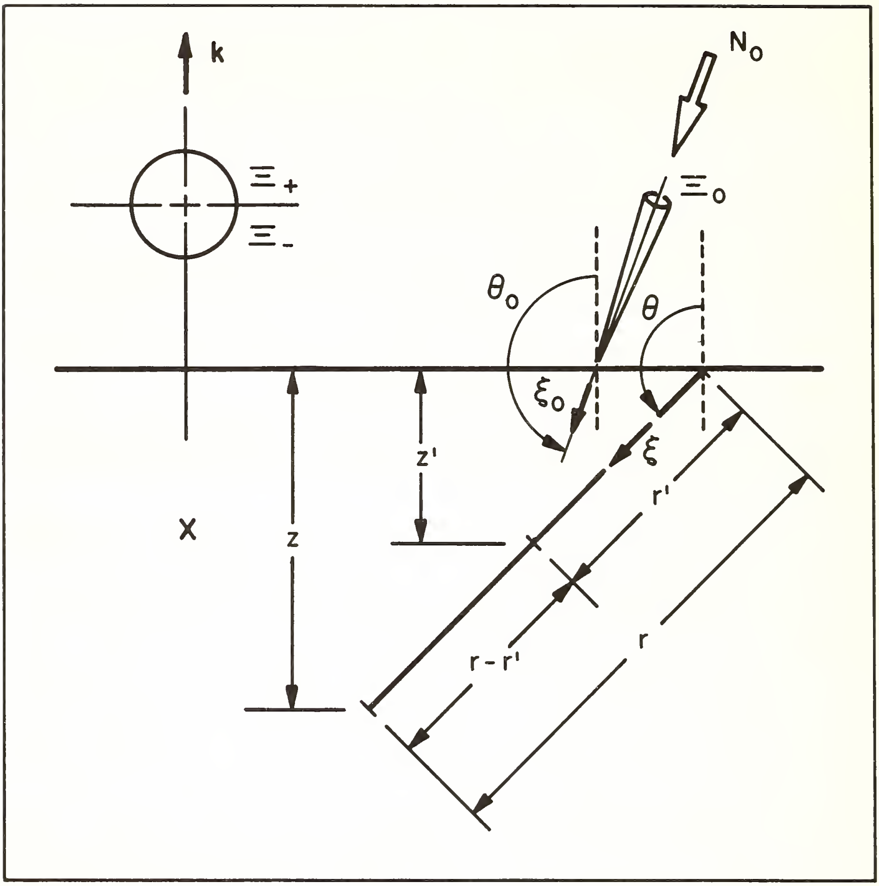

FIG. 5.2 Computing the primary scattered radiance in a hydrosol, induced by a collimated source N_o.

we have:

$$N_*^1(z,\xi) = N_o e^{\alpha z \sec \theta_o} \int_{\Xi_o} \sigma(z;\xi';\xi)\,d\Omega(\xi')$$

$$= N_o e^{\alpha z \sec \theta_o} \sigma(\xi_o;\xi)\Omega_o \qquad (7)$$

Here Ξ_o is the set of directions, of solid angle Ω_o, over which the incident beam has uniform radiance N_o. Note also that we have used the homogeneity of X in freeing σ of depth dependence. Next, we apply the path radiance operator **T** over the path depicted in Fig. 5.2:

$$N^1(z,\xi) = N_*^1 T = \int_0^r N_*(z',\xi) T_{r-r'}(z',\xi) dr'$$

$$= - \sec\theta \, N_o \, \sigma(\xi_o;\xi) \, \Omega_o \int_0^z e^{\alpha z' \sec\theta_o} \, e^{\alpha(z-z')\sec\theta} \, dz'$$

$$= - \sec\theta \, N_o \, \sigma(\xi_o;\xi)\Omega_o \, e^{\alpha z \sec\theta} \int_0^z e^{\alpha z'(\sec\theta_o - \sec\theta)} dz'$$

Therefore:

$$\boxed{N^1(z,\xi) = - \frac{N_o \sec\theta \, \sigma(\xi_o;\xi) \, \Omega_o e^{\alpha z \sec\theta_o}\left[1-e^{-\alpha z(\sec\theta_o - \sec\theta)}\right]}{\alpha(\sec\theta_o - \sec\theta)}} \qquad (8)$$

This canonical representation of $N^1(z,\xi)$, in which $\cos\theta = \xi\cdot K$, holds for all paths such that $\theta \neq \theta_o$. For the case $\theta = \theta_o$, we return to the penultimate equality and evaluate the integral anew, or use L'Hospital's rule in (8). Clearly, the new integral value is simply z for the case $\theta_o = \theta$. Comparing (8) with the canonical form, with the latter now evaluated for the case n = 1:

$$N^1(z,\xi) = \frac{N_*^1(z,\xi)}{\alpha + K^1 \cos\theta} \quad , \qquad (9)$$

we see that the following equality must hold:

$$\boxed{\alpha + K^1 \cos\theta = \frac{-\alpha(\sec\theta_o - \sec\theta)}{\sec\theta[1-e^{-\alpha z(\sec\theta_o - \sec\theta)}]}} \qquad (10)$$

From this we can, if required, solve for K^1 (which generally is a function of z,θ, and, also in the present case, the parameter θ_o). Observe that for $\sec\theta_o \geq \sec\theta$, i.e., for $\theta \leq \theta_o$.

$$\lim_{z\to\infty} K^1(z,\xi) = - \alpha\sec\theta_o$$

and for $\theta > \theta_o$,

$$\lim_{z\to\infty} K^1(z,\xi) = - \alpha\sec\theta \quad .$$

This shows directly that the K-function for primary radiance eventually, i.e., for sufficiently great depths, becomes

independent of ξ over large sets of directions (i.e., when $\theta \leq \theta_0$). This phenomenon of the eventual partial independence of K^1 with respect to direction, presages an analogous behavior of the complete K-function for observable radiance; we will study this depth behavior of K in more detail in Chapter 10.

We now summarize the main results of our illustrative example: By evaluation of (6) for the case of $n = 1$ and comparing the resultant representation of N^1 with that given by the canonical form (4), we deduce the necessary form of the K-function K^1 for N^1. The usual classical method of looking at N^1 is by means of formulas of the structure of (8). Our studies of the canonical equation of transfer in Chapter 4, extended to the present setting, now show that (8) is but a special form of the canonical equation for primary radiance N^1, as given in (9). Hence (8) may be given the compact and intuitively useful canonical form (9) provided K^1 is as given implicitly by (10).

Concluding Observations

In conclusion we note that the integrations leading to (8) may be redone now over a path $\mathcal{P}_r(z_o,\xi)$ with initial point at depth $z_o > 0$. The result will be a path radiance N_r^1 expression, the special case of N_r^n for $n = 1$, leading to an instance of (6). Observe that $N^n(z,\xi)$ in (4) and $N_r^n(z,\xi)$ in (5) are equal for every z and ξ, being but two ways of expressing the same radiance: Whereas (4) expresses the radiance $N^n(z,\xi)$ as a value of the radiance distribution $N^n(z,\cdot)$ at depth z for the direction ξ, equation (5), on the other hand, expresses the same radiance now by conceptually partitioning it into two parts associated with an arbitrary path $\mathcal{P}_r(z_o,\xi)$ in the medium. In other words, we can carry over without change from the discussions of Chapter 4 to the present setting of n-ary concepts, all interpretations of path radiance N_r^*, transmitted residual radiance N_r^o, and apparent radiance N_r, arrived at in those earlier discussions. It is of interest to emphasize in particular a powerful but simple model for radiance distributions that arises when we represent N^* rather than N by means of the general equation (2) of Sec. 4.4. For such a model "N_*" in (1) of Sec. 4.4 is replaced by "N_*^*". The correct basis for this model is (7) of Sec. 5.2.

5.4 The Natural Solution for Radiance

We return now to the main thread of the argument, begun in 5.2, leading to the development of the natural solution of the equation of transfer. Our most basic intuitions about light fields in the sea and the air and generally for any optical medium, lead us to think of the radiance perceived by our eyes and our instruments as consisting of multiply-scattered light, i.e., light which has undergone one, two, three, and generally very large numbers of scattering operations after its entrance into the medium and before its incidence on the retina or photocell located somewhere in the medium. It is natural then (hence the name of the present

mode of solution) to attempt to construct a solution of the equation of transfer for radiance by constructing all the n-ary radiance functions N^n within a given optical medium X and to sum them to obtain the requisite radiance field throughout the medium. Thus we are led to write:

$$\boxed{\text{"}N(z,\xi)\text{"} \quad \text{for} \quad \sum_{j=0}^{\infty} N^j(x,\xi)} \tag{1}$$

and hope that the function N so defined satisfies the equation of transfer. We call N defined by (1) the *natural solution of the equation of transfer.* We now show that the word "solution" in the name for N is indeed justified.

We begin by using (14) of Sec. 5.1 to write $N(x,\xi)$ in (1) as:

$$N(x,\xi) = \sum_{j=0}^{\infty} N^o \mathbf{s}^j(x,\xi)$$

or more compactly in functional form as:

$$N = \sum_{j=0}^{\infty} N^o \mathbf{s}^j$$

In this way we come to define the basic operator $\mathbf{S}$ for the natural solution, i.e., we can now write:

$$\text{"}\mathbf{S}\text{"} \quad \text{for} \quad \sum_{j=0}^{\infty} \mathbf{s}^j \quad , \tag{2}$$

where "S^o" denotes the *identity operator I*, with the property $fI = f$ for every radiance function. With this definition the natural solution representation takes the form:

$$\boxed{N = N^o \mathbf{S}} \tag{3}$$

By means of this representation, the formal verification that N in (3) is a solution of the equation of transfer is readily forthcoming via the following eight main steps:

$$N = N^o\mathbf{S} = N^o(I + \sum_{j=1}^{\infty} \mathbf{s}^j)$$

$$= N^o(I + (\sum_{j=0}^{\infty} \mathbf{s}^j)\mathbf{s}^1)$$

$$= N^o + (N^o\mathbf{S})\mathbf{s}^1$$

$$= N^o + NS^1$$

$$= N^o + (NR)T$$

$$= N^o + N_*T$$

$$= N^o + N^*$$

We have therefore shown that:

$$N = N^o + N^* \quad , \tag{4}$$

which is the integral form of the equation of transfer (re: (1) of Sec. 3.15). An alternative approach to the above demonstration is to show that N as defined by (1) is a solution of the integrodifferential equation of transfer. The basis for such a demonstration is given by (7) of Sec. 5.2. It remains only to add (2) of Sec. 5.2 to each side of (7) and reduce the results.

To summarize our findings: We have shown that the natural mode of constructing the radiance function N from the n-ary radiance functions N^n, $n \geq 0$, leads to a solution--the *natural solution*--of the equation of transfer. It also may be seen that N so constructed is a unique solution in the sense that whenever N' is also a solution of (4), then N' = N. The mathematical basis for the existence and uniqueness of the natural solution will be described in Sec. 5.12.

We conclude by observing that the natural solution of the equation of transfer is not only fundamental from an intuitive physical point of view, but that it in essence exemplifies a mode of function construction which has been of increasing importance in the logical foundations of mathematics in recent years. This mode of construction--the enumerably recursive mode of construction--is very closely related to the natural mode of construction defined above and is coming under intensive study principally because of the current strides in developing ultrafast mechanical aids to numerical and logical computations. These developments will eventually make feasible the computation of relatively high scattering orders n for N^n, so that finite sums of the form

$$N^o + N^1 + N^2 + \ldots + N^n$$

will constitute appropriately adequate approximations to the ideal natural solution N. Thus we will eventually be able to go far beyond the first order solutions

$$N^o + N^1 = N^o + \frac{N^1_*}{\alpha + K^1 \cos \theta} \tag{6}$$

(cf. (8), (9) of Sec. 5.3) to which many classical studies in atmospheric and hydrologic optics were hitherto limited because of the relatively heavy demand on manipulative skill (and time!) needed to evaluate N^2, N^3 and higher order n-ary radiance functions.

5.5 Truncated Natural Solutions for Radiance

We now investigate the effect of truncating the natural solution of the equation of transfer after a finite number of terms. While the natural solution is an ideal conceptual tool in the study of radiative transfer theory, as has been demonstrated at length in Chapter III of Ref. [251], the solution can almost never be evaluated completely either numerically or theoretically, because of the infinite number of terms comprising the solution. We are then in practice obliged to stop the accumulation of the terms after a finite number of them have been evaluated. The question then arises as to the closeness of the resultant truncated solution to the natural solution. We shall now consider this question in detail.

Throughout the remainder of this section we shall choose as our setting a source-free homogeneous plane parallel optical medium X of arbitrary depth with a steady internal light field induced by arbitrary incident radiance distributions N_o at each point of the upper boundary of the medium. The volume scattering function σ and the volume attenuation function α are otherwise arbitrary.

Now, starting with the natural solution N of the equation of transfer as defined in (1) of Sec. 5.4, we write:

$$N = \sum_{j=0}^{k} N^j + \sum_{j=k+1}^{\infty} N^j \tag{1}$$

The central question of the present discussion may now be phrased as follows. Writing:

$$"N^{(k)}" \quad \text{for} \quad \sum_{j=0}^{k} N^j \quad ,$$

we ask: by how much does the finite sum $N^{(k)}$ differ from the infinite sum N; or in other words, what is the general order of magnitude of

$$\sum_{j=k+1}^{\infty} N^j \quad ?$$

To answer this question we shall obtain an upper bound on the values of the difference $N-N^{(k)}$. This upper bound shall serve as a measure of the difference between the functions N and $N(k)$.

We begin by letting "$\bar{N}^o$" denote the upper bound of the initial radiance function N^o within X (re: (1) of Sec. 5.1). This upper bound is easily evaluated in general, and in particular in all natural hydrosols this upper bound is actually

attained by N^o at the air-water boundary of the medium. Indeed, for sunny days, N^o is almost invariably the apparent radiance of the run as seen just below the surface of the medium.

The upper bound of the primary radiance function N^1 is obtained by first 'bounding' N^1_*. Thus, starting with (6) of Sec. 5.1 in which n = 0, we have for every x in X and direction ξ in Ξ:

$$\begin{aligned} N^1_*(x,\xi) &= \int_\Xi N^o(x,\xi')\ \sigma(x;\xi';\xi)\,d\Omega(\xi') \\ &\leq \bar{N}^o \int_\Xi \sigma(x;\xi';\xi)\,d\Omega(\xi') \\ &= \bar{N}^o\ s \end{aligned} \tag{2}$$

Here "s" denotes the value of the volume total scattering function defined in (3) of Sec. 4.2. The reader will discern that it is sufficient at this stage to assume that:

$$\sigma(x;\xi';\xi) = \sigma(x;\xi;\xi')$$

for every ξ' and ξ at each point x of X, in order that we have:

$$s(x) = \int \sigma(x;\xi';\xi)\,d\Omega(\xi') \quad .$$

This is not an unusual requirement on σ (it is called a *reciprocal* condition) and is readily met by all σ from natural hydrosols. (For related conditions on σ, see Sec. 7.12.)

Next, use is made of (7) of Sec. 5.1 and the equality (2) just deduced to obtain:

$$\begin{aligned} N^1(x,\xi) &= \int_0^r N^1_*(x',\xi)\ T_{r-r'}(x',\xi)\,dr' \\ &\leq \bar{N}^o s \int_0^r T_{r-r'}(x',\xi)\,dr' \\ &= \bar{N}^o s \int_0^r e^{-\alpha(r-r')}\,dr' \\ &= \frac{\bar{N}^o s}{\alpha}\,(1-e^{-\alpha r}) \end{aligned}$$

$$\leq \bar{N}^o \rho \tag{3}$$

for every point x in X and direction ξ in Ξ; and where we have written:

$$\text{"}\rho\text{" for } s/\alpha \quad . \tag{4}$$

The ratio ρ is called the *albedo for single scattering* or more accurately the *scattering-attenuation ratio.* By our agreement in Sec. 4.2, namely that about the nonnegativity of the volume absorption function a, it follows that ρ satisfies the inequality $0 \leq \rho \leq 1$. For the present discussion we assume in particular that $0 < \rho < 1$. When we repeat the results (2) and (3), but now applied to $N^2(x,\xi)$ we obtain:

$$N^2(x,\xi) \leq \bar{N}^o \rho^2$$

for every x in X and ξ in Ξ. From this we can see a pattern emerging and we readily prove that:

$$\boxed{N^n(x,\xi) \leq \bar{N}^o \rho^n} \tag{5}$$

for every scattering order n, every point x in X and direction ξ in Ξ.

The inequality (5) is the main result needed for the determination of the upper bound for the difference $N-N^{(k)}$. Indeed, by direct computation, we have:

$$\begin{aligned}
N(x,\xi) - N^{(k)}(x,\xi) &= \sum_{j=k+1}^{\infty} N^j(x,\xi) \\
&\leq \sum_{j=k+1}^{\infty} \bar{N}^o \rho^j \\
&= \bar{N}^o \sum_{j=k+1}^{\infty} \rho^j \\
&= \bar{N}^o \rho^{k+1} \sum_{j=0}^{\infty} \rho^j \\
&= \frac{\bar{N}^o \rho^{k+1}}{1-\rho} \quad ,
\end{aligned}$$

which holds for every x in X and ξ in Ξ.

Summarizing, we may say that:

$$\boxed{N(x,\xi) - N^{(k)}(x,\xi) \leq \frac{\bar{N}^o \rho^{k+1}}{1-\rho}} \tag{6}$$

holds for every nonnegative integer k, every point x in X and direction ξ in Ξ.

As an example of the use of (6), suppose a given lake has a scattering-attenuation ratio of $\rho = 0.4$ for wavelength 550 mμ, and that $\bar{N}^0$ for that wavelength is 10^6 watts/(m^2 × steradian). We require for a particular computation that $N(x,\xi) - N^{(k)}(x,\xi)$ be not more than 10^4 watts (m^2 × steradian) for every x and ξ. What is the least scattering order k at which the natural solution must be truncated so that this condition is met? By (6) we require k such that:

$$10^4 < \frac{10^6 (0.4)^{k+1}}{1-(0.4)}$$

or that:

$$0.6 \times 10^{-2} < (0.4)^{k+1}$$

Forming an equality for the moment, we require:

$$\log_{10}(6\times10^{-3}) = (k + 1) \log_{10}(0.4)$$

This implies that to the nearest integer, k+1=6, so that k=5. Hence the truncation solution is required to be carried out to five scattering orders, at least.

A useful alternative formula to (6) is obtained by first noting that for media in which $\rho > 0$, we certainly have the maximum value $\bar{N}$ of $N(x,\xi)$ greater than the maximum value N^0 of $N^0(x,\xi)$. Then (6) implies:

$$\boxed{\frac{N(x,\xi) - N^k(x,\xi)}{\bar{N}} < \frac{\rho^{k+1}}{1-\rho}} \tag{7}$$

for every x in X and ξ in Ξ. The comparative merit of (7) over (6) consists in equation (7)'s ability to express the error of truncation in terms of a relative error, that is the error relative to the prevailing magnitude N of the light field. Hence for the medium at hand, carrying out the natural solution to five terms results in a *relative* error of less than 1 percent.

Before closing we shall examine the inequalities (5) and (6) for some insight they may yield about the relative importance of the various components of the decomposition of the natural light field. For example, (5) shows that n-ary radiances are on the whole less by a factor of ρ than (n-1)-ary radiances. Thus if $\rho = 1/2$, say, then $N^1(x,\xi)$ is on the whole, about half the magnitude of $N^0(x,\xi)$, and the magnitude of $N^2(x,\xi)$, in turn, is about half that of $N^1(x,\xi)$, and so on. Thus the overall magnitude of n-ary radiances decrease exponentially with scattering order n. Inequality (6) also shows that for small ρ (near 0), a given n-ary radiance varies directly as the nth power of ρ, whereas for large ρ (near 1), the n-ary radiances vary essentially hyperbolically

with $1-\rho$, i.e., as $1/(1-\rho)$. Similar observations can be made using (6) or (7). We shall return to the matter of truncated natural solutions in the following section and reconsider them for transient light fields. The reader wishing radiance bounds in a slightly more general steady state case than that considered in this section, may consult Sec. 22 of Ref. [251].

5.6 Optical Ringing Problem. One-Dimensional Case

The object of this section is to formulate the optical ringing problem in the context of radiative transfer theory and to indicate how the natural mode of solution may be used to solve the problem. In order to explain the ideas behind the optical ringing problem and its natural mode of solution without too many geometrical complications, we consider first the one-dimensional case of the problem. The three-dimensional case will be discussed in the following section.

The term, "optical ringing" has an analogous meaning to the term "reverberation" as used in the theory of sound. In fact the well-known term "reverberate" applies in principle equally to optical and acoustical phenomena. However, until recently, the relative difficulty of producing and recording optical reverberation because of the immeasurably short periods of time involved has given the acoustical discipline almost exclusive use of the term. We can use the popular acoustical meaning of the term "reverberation" to give the following nontechnical definition of the phenomenon at hand: *Optical ringing* in an optical medium is the optical reverberation of the medium set up by a narrow short pulse of monochromatic light. Hence the appropriate acoustical analogy to optical ringing would be the reverberation set up by a directional, short clap of one-note thunder. In more technical parlance the *optical ringing problem* in a medium X is the problem of determining at time $t \geq 0$, the time-dependent radiance function over X which is the solution of the equation of transfer, given a directional, spatial, and temporal Dirac-delta function input of radiance to the medium at time $t = 0$. This problem has applications to the description of time-dependent radiance fields set up by laser beams with their characteristic high power, narrow-beam, short-pulse shafts of monochromatic radiant flux. While interest in the optical ringing problem has reawakened because of the advent of the laser, it should be noted that the problem is a venerable one in radiative transfer theory and neutron transport theory, and was first studied purely for its intrinsic interest and as a fundamental block on which to build solutions with arbitrary initial time-varying, inputs (see, e.g., [211], [235], [236]).

Geometry of the Time-Dependent Light Field

The formulation of the time-dependent radiant flux problem in an optical medium X will be facilitated by finding an efficient means of depicting the space-time disposition of the radiant flux throughout the optical medium. We shall now construct such a means. In the present discussion the medium X is one-dimensional and is represented in Fig. 5.3(a)

as a line segment. We shall consider the medium to extend indefinitely on either side of the origin point 0 of the medium, with distance measured as positive toward the right.

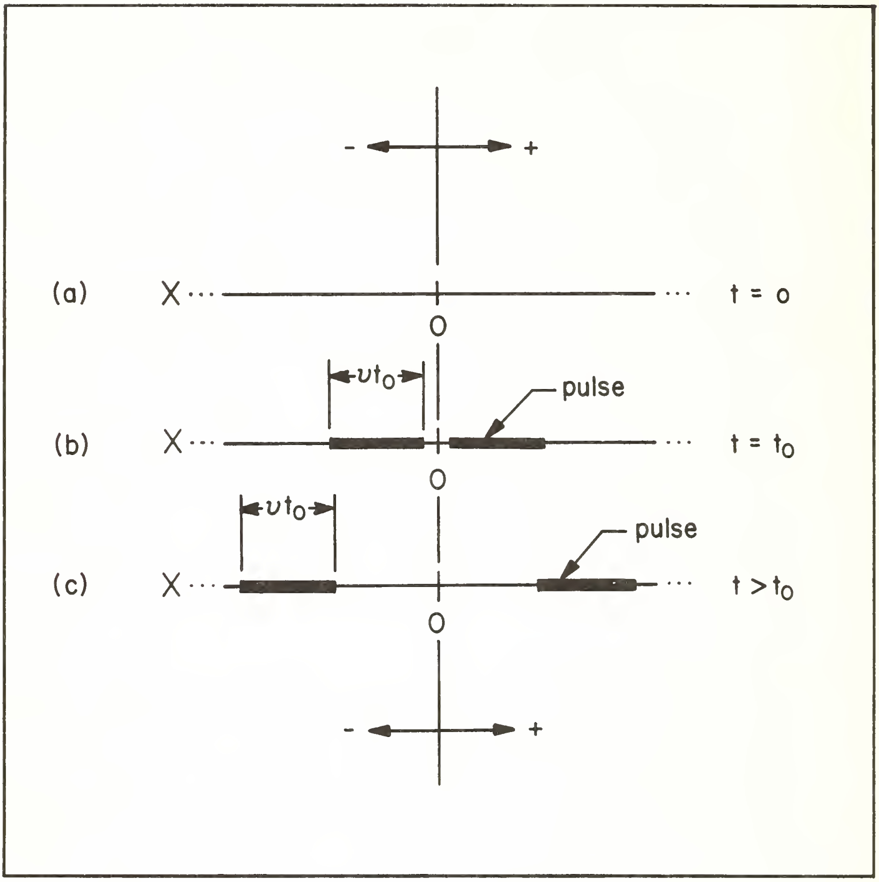

FIG. 5.3 Positions of a finite light pulse along a one-dimensional medium.

Now suppose that point 0 becomes a source of radiant flux starting at time $t = 0$ and that 0 continues to emit flux in an arbitrary fashion in both directions about 0 until time $t = t_o$, at which time the source at 0 is shut off. Let "$N_o(0,t,+)$" and "$N_o(0,t,-)$" denote these radiances of 0 at time t in the positive and negative directions, respectively. Figure 5.3(b) shows the position of the pulse emitted by 0 just after time t_o. The pulse is speeding away from point 0 into the medium on either side of 0. Figure 5.3(c) shows the position of the pulse some time later than t_o. Figures 5.3(a) through 5.3(c) are like three snapshots of the medium

X at three separate instants of time subsequent to the emission of the pulse. It would be quite instructive if instead of still shots of X at discrete time instants, we could have a moving picture of the pulse as it moves out into X from 0 and generates the field of scattered light within X. Such a means of communication is obviously unfeasible for the present work. However, an alternate and in some ways superior means of visualizing the time-dependent light field in X consists in a static space-time diagram of the pulse in X of the kind depicted in Fig. 5.4.

The description of the pulse of radiant flux from point 0 becomes relatively simple when given in terms Fig. 5.4. The space-time portrait of the pulse is given by the shaded V-shaped region in the space-time diagram. To find the instantaneous position of the pulse in X at time t', first go along the time axis erected perpendicular to X until time

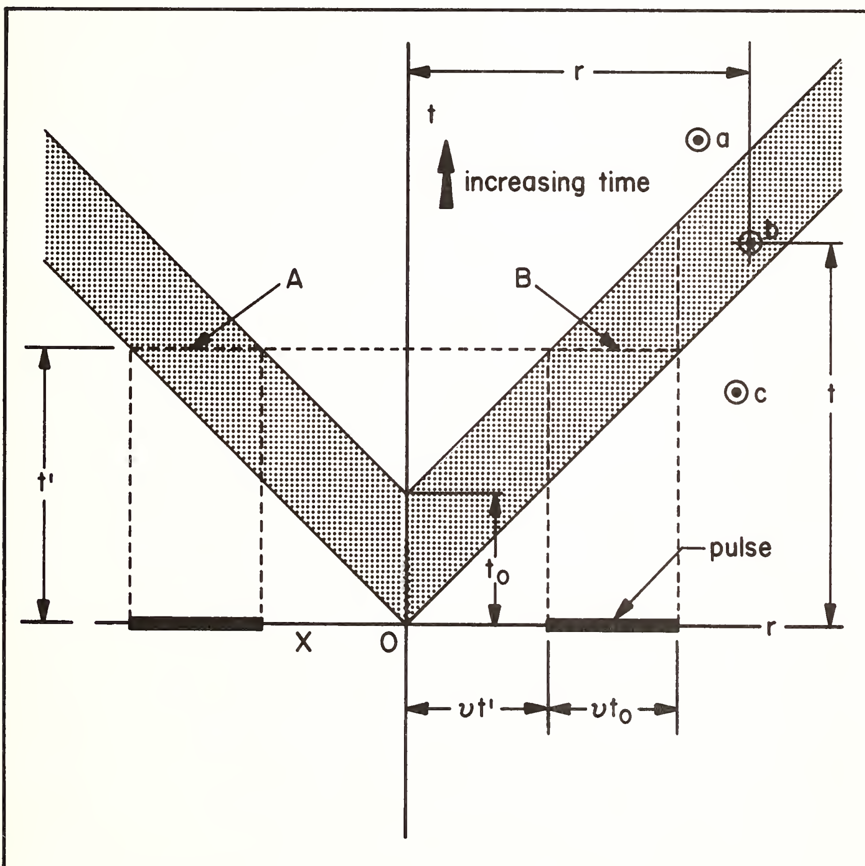

FIG. 5.4 A space-time portrait of the pulse in Fig. 5.3. The world region of the pulse is shaded.

point t' is reached. Then draw a straight line through t' parallel to X. This line will intersect the shaded region in generally two segments A and B. The perpendicular projection of these segments down onto X will then give the location of the pulse in X at time $t' \geq 0$. The slope of any straight line segments parallel to the boundaries of the shaded region of the pulse are such that, as t' units of the time axis are traversed, vt' units of the space axis are traversed, where v is the speed of light in X. We assume v to be constant over X. The shaded region of Fig. 5.4 is called the *world region* of the pulse.

It follows from the axioms of special relativity that, relative to the frame at 0, the space-time line traced out by a material particle in X cannot have an arbitrary slope, but rather one which is bounded as follows. If r(t) is the distance of the particle from 0 at time t, then:

$$\left| \frac{dr(t)}{dt} \right| \leq v$$

for every t for which r(t) is defined in the frame anchored at 0. In particular, the slopes of the *world lines* (i.e., space-time trajectories) of the photons comprising the pulse of light from 0 are exactly of magnitude v, with respect to the time axis. Thus on the one hand, the world line of a particle stationary in X is a vertical line, and on the other hand, that of a photon is parallel to one of the boundary lines of the shaded region in Fig. 5.4. All naturally moving particles in X must therefore have the tangents to their world lines always between (or coincident) with these two extremes, with respect to the r,t frame of reference at 0.

The space-time diagram also aids in visualizing the various possibilities of radiometric interactions between points of X. Thus, points a, b, and c in Fig. 5.4 depict the three possible dispositions of points in space time with respect to the pulse from 0. Point b(=(r,t)) is in the world region of the pulse, and so represents a point of X at distance r from 0 which at time t is being irradiated by radiant flux comprising some of the pulse from 0. Points a and c on the other hand are not in the world region of the pulse. Point a in particular represents a point in X *after* the pulse has gone by it (to find the contemporaneous pulse to a, draw a horizontal line through a, and the segment it determines with the world region of the pulse is the requisite position of the pulse). Point c represents a point in X *before* the pulse has gone by it. Points a and c thus have the property in common that they do not lie on the world region of the pulse from 0; however, points a and c differ from one another in a fundamental sense. Indeed, the point in X corresponding to a may eventually feel the effects of the pulse through scattering of flux from the pulse; however, the point in X corresponding to point c in the space time plane is "forever" immune to the direct or indirect effects of the pulse. Here we are implicitly adopting another empirical fact of macroscopic physics: Effects of an event may propagate futureward in space-time but not pastward. When this fact is combined with that about the limits on the slopes of the world lines

of particles mentioned above, we can readily delimit those parts of the space-time plane over (or through) which they can effect or be effected by a given event (represented as a point) in the plane. These regions are shown in Fig. 5.5(a) for an arbitrary point a. In general, for two points a and b in the space-time diagram associated with X, the common region of possible interaction is the shaded intersection of the futureward sector of b with the pastward sector of a, as shown, in Fig. 5.5(b). If the intersection region is empty, then the two points cannot interact.

With these preliminary observations in mind, we may now use the the general space-time diagram to help in the study of the time-dependent radiant flux problem on X. Starting

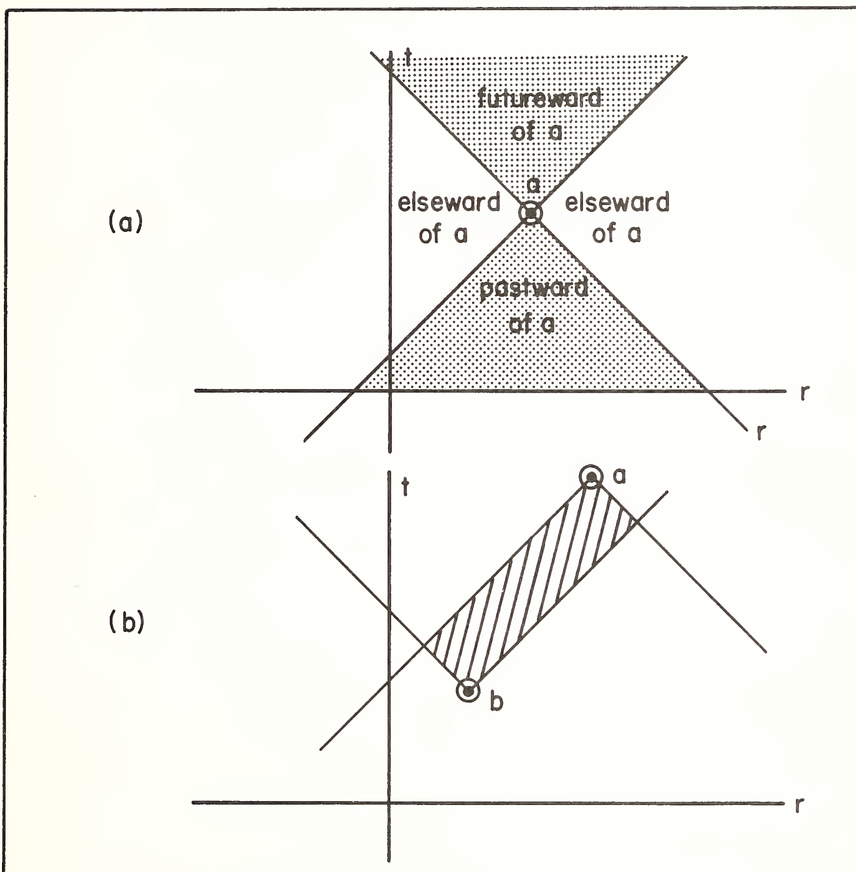

FIG. 5.5 Part (a) depicts those points of space-time about point a which lie in a's future, past, and elsewhere from a. Part (b) shows the common region (shaded) shared by the future cone of b and the past cone of a. When this shaded region exists, then b can send a light signal to a.

with a fresh space-time diagram of the pulse emitted by point 0 in X, as in Fig. 5.6, we see that the pulse effects at time t at some point a distance r from 0 in the medium arrive through the pastward sector of the point (r,t). In particular, the region of X contributing scattered flux of all orders to (r,t) is bounded by a(r,t), b(r,t), where we have written:

$$"a(r,t)" \quad \text{for} \quad (r-vt)/2 \tag{1}$$

$$"b(r,t) \quad \text{for} \quad (r+vt)/2 \tag{2}$$

For example, if r = 0, then the interaction region of X at each time t is an interval on X of length vt centered on 0. The route of radiant flux from 0 to point (r,t) may be quite devious. Two sample routes from 0 to (r,t) are shown by the

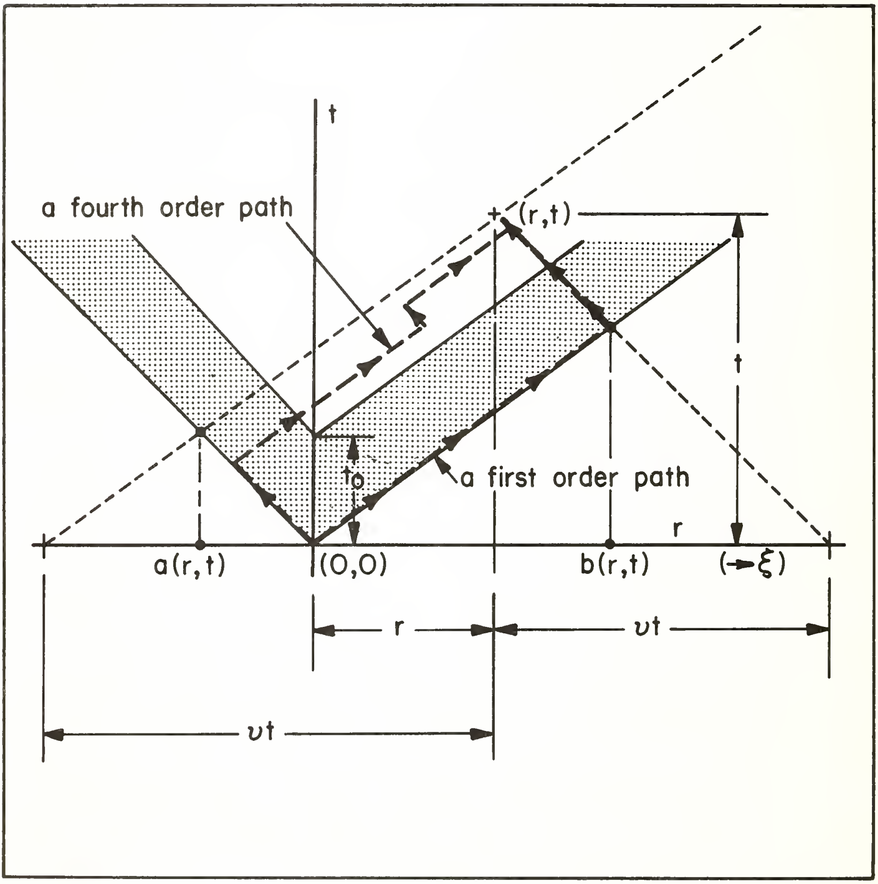

FIG. 5.6 Computing the scattered light reaching space-time point (r,t) after starting from the origin (0.0).

dashed lines in Fig. 5.6. In one of the cases the flux reaching (r,t) is intended to be fourth order radiant flux. The spatial component of the path taken by this sample of radiant flux is obtained by projecting the space-time path onto X. Observe that in this particular example the only way radiant flux can reach (r,t) from 0 is by undergoing at least one back scattering operation.

The Equation of Transfer

The integral form of the equation of transfer for the one-dimensional optical medium X defined above will now be derived. Before going into the details, however, it may be well to reemphasize that the significance of a one-dimensional optical medium lies not so much in its power to represent an actual physical setting as it does in its ability to depict with a minimum of geometric complication the essential algebraic structures of the associated three-dimensional problem. Therefore, the resultant equation of transfer derived below for the present one-dimensional setting will, in all its algebraic essentials, be representative of the full three-dimensional case, but will not be encumbered with details arising from the latter's relatively complex geometrical structure. These details will be faced in the following section.

Under suitably adapted definitions of the radiance function and inherent optical properties for X, the equation of transfer for the one-dimensional optical medium X follows formally from the integral form of (4) of Sec. 3.15. In this way we extend the logical chain from the interaction principle of Chapter 3 to the present radiative transfer discussion. In particular the present equation of transfer is obtained by postulating the characteristic form of the volume scattering function for one-dimensional media:

$$\sigma(x;\xi';\xi,t) = \rho(x,t)\ \delta(\xi+\xi') + \tau(x,t)\ \delta(\xi'-\xi)$$

where ξ is one of the two directions $(\pm\xi)$ along the medium, and δ is the well-known Dirac-delta function. The functions ρ and τ are, respectively backward and forward scattering functions for X. Furthermore, the values of the radiance function are now of the form $N(x,t,+)$ or $N(x,t,-)$, where "+" and "-" denote flux in the direction + or -, respectively. That is, we have written:

$$\text{"}N(x,\xi',t)\text{"} \quad \text{for} \quad N(x,t,+)\ \delta(\xi'-\xi) + N(x,t,-)\ \delta(\xi'+\xi)$$

Since the points x in X are located by one number only, namely the signed distance r from 0 to x, we will write "r" in place of "x" throughout the one-dimensional setting. It now follows from (8) of Sec. 3.14 with the adopted form of σ and N (and assuming here only that δ is idempotent, i.e., $\delta^2 = \delta$, at least formally) that the path function values $N_*(r,t,\pm)$ associated with directions $\pm$ are:

$$N_*(r,t,+) = N(r,t,+)\ \tau(r,t) + N(r,t,-)\ \rho(r,t) \qquad (3)$$

$$N_*(r,t-) = N(r,t,-)\ \tau(r,t) + N(r,t,+)\ \rho(r,t) \qquad (4)$$

The time-dependent integral form of the equation of transfer for the one-dimensional case therefore consists of the following two equations (one for each direction (+,-):

$$N(r,t,+) = u(r)N_o(0,t - |r/v|, +)\ T_r + \int_{a(r,t)}^{r} N_*(r',t',+)T_{r-r'}\,dr' \qquad (5)$$

$$N(r,t,-) = u(-r)N_o(0,t-|r/v|, -)\ T_{-r} + \int_{r}^{b(r,t)} N_*(r',t',-)T_{r'-r}\,dr' \qquad (6)$$

where $u(r) = 1$ if $r \geq 0$, and $u(r) = 0$ if $r < 0$. All terms except the transmittance terms in these two equations have been defined in the present section. The transmittances are represented as in (3) of Sec. 3.11; thus for the present case we have:

$$"T_{s-r}" \quad \text{for} \quad \exp\left\{-\int_r^s \alpha\, dr'\right\}$$

in which matters are arranged so that $r \leq s$.

Operator Form of the Equation of Transfer

We next cast the pair of transfer equations (5), (6) into an operator form which at once suggests the appropriate instance of the natural solution for the present case. Thus, we agree to write:

$$"N^o_+(r,t)" \quad \text{for} \quad u(r)N_o(0,t-|r/v|,+)\ T_r$$

$$"N^o_-(r,t)" \quad \text{for} \quad u(-r)N_o(0,t-|r/v|,-)\ T_{-r}$$

and further, we write:

$$"T_+" \quad \text{for} \quad \int_{a(r,t)}^{r} [\ ]\tau T_{r-r'}\,dr'$$

$$\text{"}R_-\text{"} \quad \text{for} \quad \int_{a(r,t)}^{r} [\ ]\rho T_{r-r'}\, dr'$$

$$\text{"}T_-\text{"} \quad \text{for} \quad \int_{r}^{b(r,t)} [\ ]\tau T_{r'-r}\, dr'$$

$$\text{"}R_+\text{"} \quad \text{for} \quad \int_{r}^{b(r,t)} [\ ]\rho T_{r'-r}\, dr'$$

With these assignations, (5), (6) become:

$$N(r,t,+) = N_+^0(r,t) + NT_+(r,t) + NR_-(r,t)$$

$$N(r,t,-) = N_-^0(r,t) + NT_-(r,t) + NR_+(r,t)$$

The notation "$NT_+(r,t)$", e.g., denotes the value of the function NT_+ at (r,t), and NT_+ is the result of acting on N with the operator T_+. These equations may be made more compact and at the same time more algebraic in appearance by writing:

$$\text{"}N_+\text{"} \quad \text{for} \quad N(\cdot,\cdot,+)$$

$$\text{"}N_-\text{"} \quad \text{for} \quad N(\cdot,\cdot,-)$$

$$\text{"}N_+^0\text{"} \quad \text{for} \quad N_+^0(\cdot,\cdot)$$

$$\text{"}N_-^0\text{"} \quad \text{for} \quad N_-^0(\cdot,\cdot) \quad .$$

With these abbreviations for the four radiance functions we then can write (5) and (6) as:

$$\boxed{\begin{aligned} N_+ &= N_+^0 + N_+T_+ + N_-R_- \\ N_- &= N_-^0 + N_-T_- + N_+R_+ \end{aligned}} \qquad \begin{aligned} &(7) \\ &(8) \end{aligned}$$

This form of the equation of transfer now suggests that we write:

$$\text{"S"} \quad \text{for} \quad \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix} \tag{9}$$

along with:

$$\text{"N"} \quad \text{for} \quad (N_+, N_-) \tag{10}$$

and

$$\text{"}N^o\text{"} \quad \text{for} \quad (N_+^o, N_-^o) \tag{11}$$

so that the system (7) and (8) written in vector notation becomes:

$$(N_+ N_-) = (N_+^o, N_-^o) + (N_+, N_-)S \tag{12}$$

or, succinctly:

$$\boxed{N = N^o + NS} \tag{13}$$

In this way we have reattained the basic structure of the integral equation of transfer, now for the simple one-dimensional context (recall, e.g., the derivation of (4) of Sec. 5.4). It follows that we may at once apply the natural solution procedure to (13) and thereby compute directly the scattering order components of N to as great a degree of accuracy as desired. This will now be done.

The Natural Solution

Starting with equation (13), and treating N as if it were an unknown in a simple linear algebraic equation we obtain:

$$N = N^o(I-S)^{-1}$$

where $(I-S)^{-1}$ may be shown to be expandable into an infinite series:

$$(I-S)^{-1} = I+S+S^2+S^3+\ldots \tag{14}$$

We have encountered such a type of expansion several times before in the present work. For instance it was used in Example 15 of Sec. 2.11, and it occurred many times in the examples of Chapter 3. Finally, closely related series were encountered earlier in this chapter (see (2) of Sec. 5.4). Hence the requisite solution of the time-dependent equation of transfer for the one-dimensional optical medium takes the form:

$$\boxed{(N_+,N_-) = \sum_{j=0}^{\infty} (N_+^o,N_-^o)S^j} \tag{15}$$

An Example

As an illustration of the natural solution for the present one-dimensional optical ringing problem suppose the medium X is homogeneous and in the steady state, so that ρ and τ are constant valued functions over space and time. Suppose further that N_+^o and N_-^o are each constant valued and over a time period from $t = 0$ to $t = t_o > 0$ (a slight simplification occurs if these are of Dirac-delta temporal structure; however, a temporally finite pulse, is at present a more useful and realistic input for X, and accordingly is adopted). Then, carrying out the expansion (15) to second order scattering, we have:

$$(N_+,N_-) = (N_+^o,N_-^o) + (N_+^o,N_-^o)S + (N_+^o,N_-^o)S^2 \tag{16}$$

Since

$$S^2 = \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix}^2 = \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix} \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix}$$

$$= \begin{bmatrix} T_+^2+R_+R_- & T_+R_++R_+T_- \\ R_-T_++T_-R_- & R_-R_++T_-^2 \end{bmatrix} ,$$

we have from (16) for the first component N_+ of the vector (N_+N_-):

$$N_+ = N_+^o + \left[N_+^oT_+ + N_-^oR_-\right] + \left[N_+^o(T_r^2 + R_+R_-) + N_-^o(R_-T_+ + T_-R_-)\right] \tag{17}$$

and for the second component N_- of the vector (N_+,N_-):

$$N_- = N_-^o + \left[N_+^oR_+ + N_-^oT_-\right] + \left[N_+^o(T_+R_+ + R_+T_-) + N_-^o(R_-R_+ + T_-^2)\right] \tag{18}$$

Equations (17) and (18) show how the natural solution (15) can be constructed order by order for an evolution of (N_+,N_-). If still another scattering order is needed, we include S^3:

$$S^2S = \begin{bmatrix} T_+^2 + R_+R_- & T_+R_+ + R_+T_- \\ R_-T_+ + T_-R_- & R_-R_+ + T^2 \end{bmatrix} \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix}$$

$$= \begin{bmatrix} T_+^3 + R_+R_-T_+ + T_+R_+R_- + R_+T_-R_- & T_+^2R_+ + R_+R_-R_+ + T_+R_+T_- + R_+T_-^2 \\ R_-T_+^2 + T_-R_-T_+ + R_-R_+R_- + T_-^2R_- & R_-T_+R_+ + T_-R_-R_+ + R_-R_+T_- + T_-^3 \end{bmatrix}$$

To show how the second order operators in (17) and (18) are applied in practice, let us assume explicitly that $N_o(0,t,-) = 0$ for all t, and that N is the constant value of the radiance pulse $N_o(0,t,+)$ of duration t_o, starting at $t = 0$, in the direction ξ, i.e., of increasing r. The present situation then constitutes an approximate model of the light field generated by a laser-like beam pulse of duration t_o seconds in the immediate vicinity of the beam. The outgoing field N_+ evaluated at $r = 0$ for every $t \geq 0$ is then, according to (17):

$$N(0,t,+) = N_o(0,t,+) + N_+^oT_+(0,t) + N_+^o(T_+^2 + R_+R_-)(0,t) \tag{19}$$

The incoming field N_- evaluated at $r = 0$ for every $t \geq 0$ is, according to (18):

$$N(0,t,-) = N_+^oR_+(0,t) + N_+^o(T_+R_+ + R_+T_-)(0,t) \tag{20}$$

In each of these equations, we have $N_o(0,t,+) = N$ for $0 \leq t \leq t_o$ and $N_o(0,t,+) = 0$ for every other t.

Let us consider (20) in more detail. The first order scattering term, unraveled, becomes:

$$N_+^oR_+(r,t) = \int_r^{b(r,t)} N_+^o(r',t')\, \rho(r',t')T_{r'-r}\, dr' , \tag{21}$$

in which we are to set $r = 0$, and $t' = t-|r'/v|$. A study of part (a) of Fig. 5.7, which depicts the present situation, and a study of the definitions N_+^o and N_-^o, shows that this integral is best evaluated by establishing two cases: Case (i) pertains whenever $t \leq t_o$; Case (ii) pertains whenever $t > t_o$. The particular forms of (21) for these two cases are as follows. Case (i), ((0,t) in the pulse):

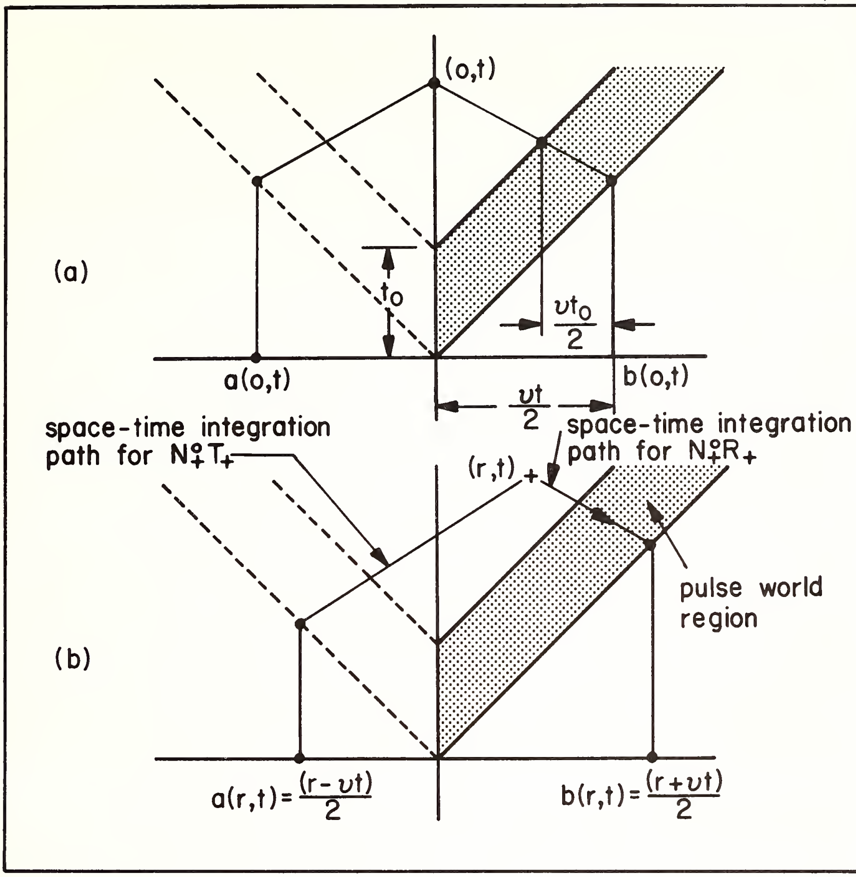

FIG. 5.7 Space-time path integration details.

$$N_+^o R_+(0,t) = N\rho \int_0^{vt/2} e^{-2\alpha r'}\, dr'$$

$$= \frac{N\rho}{2\alpha}\left(1 - e^{-\alpha vt}\right) \tag{22}$$

Case (ii), ((0,t) after the pulse):

$$N_+^oR_+(0,t) = N\rho \int_{(vt/2)-(vt_o/2)}^{vt/2} e^{-2\alpha r'}\, dr'$$

$$= \frac{N\rho\, e^{-\alpha vt}}{2\alpha}\left[e^{\alpha vt_o} -1\right] \tag{23}$$

Equations (22) and (23) describe the first order scattered radiance flowing in the negative direction of X, at $r = 0$.

For the radiance at a general space-time (r,t), we once again require two cases: Case (i) pertains when $(t-t_o)v \le |r| \le vt$; and Case (ii) pertains when $|r| < (t-t_o)v$. These cases reduce to the special instances considered above when $r = 0$. In general, Case (i) holds when the space-time point (r,t) is in the world region of the pulse; Case (ii) holds when (r,t) is futureward (above or after) the world region of the pulse. Returning now to (21), we evaluate it for a general point (r,t), according to the two cases ((b) of Fig. 5.7): Case (i), ((r,t) in the pulse):

$$N^oR_+(r,t) = N\rho \int_{r}^{(r+vt)/2} e^{-\alpha r'}\, e^{-\alpha(r'-r)}\, dr'$$

$$= \frac{N\rho}{2\alpha}\left[e^{-\alpha r} - e^{-\alpha vt}\right] \tag{24}$$

Case (ii), ((r,t) after the pulse):

$$N_+^oR_+(r,t) = N\rho \int_{\frac{(r+vt)}{2}-\frac{vt_o}{2}}^{(r+vt)/2} e^{-\alpha r'}\, e^{-\alpha(r'-r)}\, dr'$$

$$= \frac{N\rho e^{-\alpha vt}}{2\alpha}\left[e^{\alpha vt_o} -1\right] \tag{25}$$

Equations (24) and (25) describe the primary scattered radiance in the direction $-\xi$ in X at a general space-time point (r,t) such that $r \le vt$. For $r > vt$, the primary radiance is clearly zero, as may be seen by reviewing the geometry of the space-time plane discussed earlier. Furthermore, this value is approached by (24) as (r,t) approaches the lower boundary of the pulse's world region, i.e., the line defined by $r = vt$. Hence $N_+^oR_+$ is uniquely defined throughout the whole space-time diagram.

We turn next to illustrate the evaluation of the second order scattering terms in (20). We first consider $N_+^oT_+R_+$. This is interpreted to be the result of the operation of R_+

on $N_+^oT_+$. The latter, in turn, gives the primary scattered radiance in the direction + for a general space-time point (r,t):

$$N_+^oT_+(r,t) = \int_{a(r,t)}^{r} N_+^o(r',t')\ \tau(r',t')\ T_{r-r'}\ dr'$$

in which we are to set $t' = t-r'/v$. A study of Fig. 5.7 shows that, for the present source condition, we have $N_+^o(r',t') = 0$ for $r' < 0$ (no source radiant flux in the direction + at any time for points $r' < 0$). Hence the integration may begin at $r' = 0$, instead of $a(r,t)$ $(=(r-vt)/2)$. Furthermore, $\tau(r',t')$ is constant of fixed value τ for all r' and t'. Hence, Case (i), ((r,t) in the pulse):

$$N_+^oT_+(r,t) = N\tau \int_o^r e^{-\alpha r'}\ e^{-\alpha(r-r')}\ dr'$$

Hence:

$$N_+^oT_+(r,t) = N\tau r\ e^{-\alpha r} \tag{26}$$

Case (ii), ((r,t) after the pulse):

$$N_+^oT_+(r,t) = 0 \tag{27}$$

Equations (26) and (27) give the primary scattered radiance in the direction $+\xi$ at a general space-time point (r,t) futureward of the origin (0,0).

We are now ready to evaluate the second order terms. Thus we have, Case (i), ((r,t) in the pulse):

$$N_+^oT_+R_+(r,t) = \int_r^{(r+vt)/2} N_+^oT_+(r',t')\ \rho(r',t')\ T_{r'-r}\ dr'$$

$$= N\tau\rho \int_r^{(r+vt)/2} r'\ e^{-\alpha r'} \cdot e^{-\alpha(r'-r)}\ dr'$$

$$= N\tau\rho\ e^{\alpha r} \int_r^{(r+vt)/2} r'\ e^{-2\alpha r'}\ dr'$$

$$= \frac{N\tau\rho}{4\alpha^2} \left\{ e^{-\alpha r}[1+2\alpha r] - e^{-\alpha vt}[1+\alpha(r+vt)] \right\} \tag{28}$$

Case (ii), ((r,t) after the pulse):

$$N_+^oT_+R_+(r,t) = \int_{\frac{(r+vt)}{2}-\frac{vt_o}{2}}^{\frac{(r+vt)}{2}} N_+^oT_+(r',t')\ \rho(r',t')\ T_{r'-r}\ dr'$$

$$= N\tau\rho\ e^{\alpha r}\int_{\frac{(r+vt)}{2}-\frac{vt_o}{2}}^{\frac{(r+vt)}{2}} r'\ e^{-2\alpha r'}\ dr'$$

$$= \frac{-N\tau\rho}{4\alpha^2}\ e^{-\alpha vt}\left\{[1+\alpha(r+vt)] - e^{\alpha vt_o}[1+\alpha(r+vt)-\alpha vt_o]\right\}$$

(29)

The final term in the second order expansion of N(0,t,-) as given in (20) is $N_+^oR_+T_-$, that is, the result of operating on $N_+^oR_+$ with T_-. Once again it is convenient to consider two cases: Case (i), ((r,t) in the pulse):

$$N_+^oR_+T_-(r,t) = \int_r^{b(r,t)} N_+^oR_+(r',t')\ \tau(r',t')\ T_{r'-r}\ dr'$$

$$= \frac{N\rho\tau}{2\alpha}\int_r^{b(r,t)} [e^{-\alpha r'} - e^{-\alpha vt'}]\ e^{-\alpha(r'-r)}\ dr'$$

$$= \frac{N\rho\tau e^{\alpha r}}{2\alpha}\int_r^{\frac{(r+vt)}{2}} \left[e^{-2\alpha r'} - e^{-\alpha vt'}\right]\ dr'$$

$$= \frac{N\rho\tau}{4\alpha^2}\left\{e^{-\alpha r} - e^{-\alpha vt} + \alpha e^{\alpha(r-vt)}[r-vt]\right\} \quad (30)$$

Case (ii), ((r,t) after the pulse):

$$N_+^o R_+ T_-(r,t) = \int_r^{b(r,t)} N_+^o R_+(r',t')\, \tau(r',t')\, T_{r'-r}\, dr'$$

$$= \int_r^{b(r,t) - \frac{vt_o}{2}} N_+^o R_+(r',t')\, \tau(r',t')\, T_{r'-r}\, dr' +$$

$$+ \int_{b(r,t) - \frac{vt_o}{2}}^{b(r,t)} N_+^o R_+(r',t')\, \tau(r',t')\, T_{r'-r}\, dr'$$

(31)

The integration in Case (ii) is shown split into two parts: that part of the integration over the segment of the space-time path after the pulse, and that over the segment of the space-time path in the pulse. The result of an integration over the futureward region of the pulse is in general not zero for secondary and higher order scattering.

The first integral in (31) uses Case (ii) for $N_+^o R_+$ evaluated in (25), and the second integral uses Case (i) above by replacing the lower limit in (30) by $(r+vt/2) - (vt_o/2)$. The requisite value N(0,t,-) is now obtained by setting $r = 0$ in the appropriate cases in (24), (25), (28), (29), (30), (31) and adding the appropriate terms, in accordance with (20).

Concluding Observations

We have carried the evaluation of N(0,t,-) far enough to show the essentials of the natural solution procedure for the one-dimensional time-dependent problem. It should be particularly noted how each step builds on the preceding step and--manipulative difficulties aside--how each step is in principle directly constructable in a finite number of operations using elementary calculus. With the advent of ever more sophisticated symbolic manipulation programs for general purpose electronic computers, it should eventually be possible to have a program which would permit the *symbolic term-by-term integration of the natural solution series* (15). We have carried the solution of the present problem far enough to show that only integrals of the type

$$\int_b^c r^n e^{-ar}\, dr$$

will be encountered in the natural solution for one-dimensional time-dependent radiative transfer problems on homogeneous spaces. With such general information a program should in principle be possible which combines simple algebraic and calculus manipulations, and which will give the two components of the nth term of (15) mechanically and relatively quickly. By having the machine run out several more terms than the second order, obtained so laboriously above, a trained human looking at the emerging terms could perhaps discern a pattern in this (or subsequently more complex problems) and thereby prepare for an inductive leap to the general term of the series. The advantages of *symbolic* over numerical integration are obvious. The former is exact at each stage whereas the latter is plagued by cumulative round-off errors. Once a symbolic integration has been performed, it may then be evaluated for the particular numerical case of interest.

One final observation can be made about the natural solution of one-dimensional time-dependent problems. This concerns extension of the analogy between the class of acoustical and optical reverberations, or as they are more commonly called, "electrical circuit transients." By studying the Laplace transform techniques of solving the problems of transients in electrical circuits (see, e.g., Chapter IX of Ref. [39]), one sees the possibility of interpreting certain terms in the final solution as analogous to the nth order scattering terms developed above. This suggests the possibility of a thoroughgoing theory, built along natural-solution lines, which should underlie and unify the particular ringing problems in the fields of optics, acoustics, transmission-line theory and electromagnetics. Mathematicians can view this as extensions of the Neumann series to space-time linear settings. An approach to such a unification can be based on the formalities developed in the present chapter since many of the operator equations appearing here are clearly interpretable in terms of the concepts of each of the preceding fields.

5.7 Optical Ringing Problem. Three-Dimensional Case

We examine next how the natural mode of solution of the equation of transfer can be applied to the problem of determining the time-dependent radiance field in a natural optical medium. The program to be followed here is that which systematically generalizes the developments of Sec. 5.1 to the time-dependent case; in particular the generalizations of the **R** and **T** operators will be the key steps in the present discussion. We begin by introducing an important geometrical concept connected with the time-dependent problem.

The Characteristic Ellipsoid

Let x and y be two points in an extensive natural optical medium X. Suppose that at time t = 0, a spherical pulse of light is emitted from x. This pulse expands about x as center and at time r/v passes point y, where r is the distance from x to y. Here v is the speed of light in X, assumed independent of location and time throughout this discussion. Just after the wave front of the pulse passes y, a

multiply-scattered radiant flux field is generally incident on y from all directions about y. We now ask: What is the region of points in X which can send radiant flux to y at an arbitrary time $t \geq r/v$? It is easy to see that at exactly $t = r/v$, this region is the straight line segment between x and y. Any points x of X off this line segment could not send scattered flux to y because the detour, however, slight, would delay the scattered flux's arrival time at y. For times t of arrival at y such that $t > r/v$, such detours are possible to some extent. The region in which the scattering detours are possible and which allow arrival at y at time t is generally an ellipsoid of revolution with x and y as foci. This may be seen by studying Fig. 5.8, and recalling that definition of an ellipsoid which characterizes it as the locus of points z such that the sum of distances $d(x,z) + d(z,y)$ is a constant.

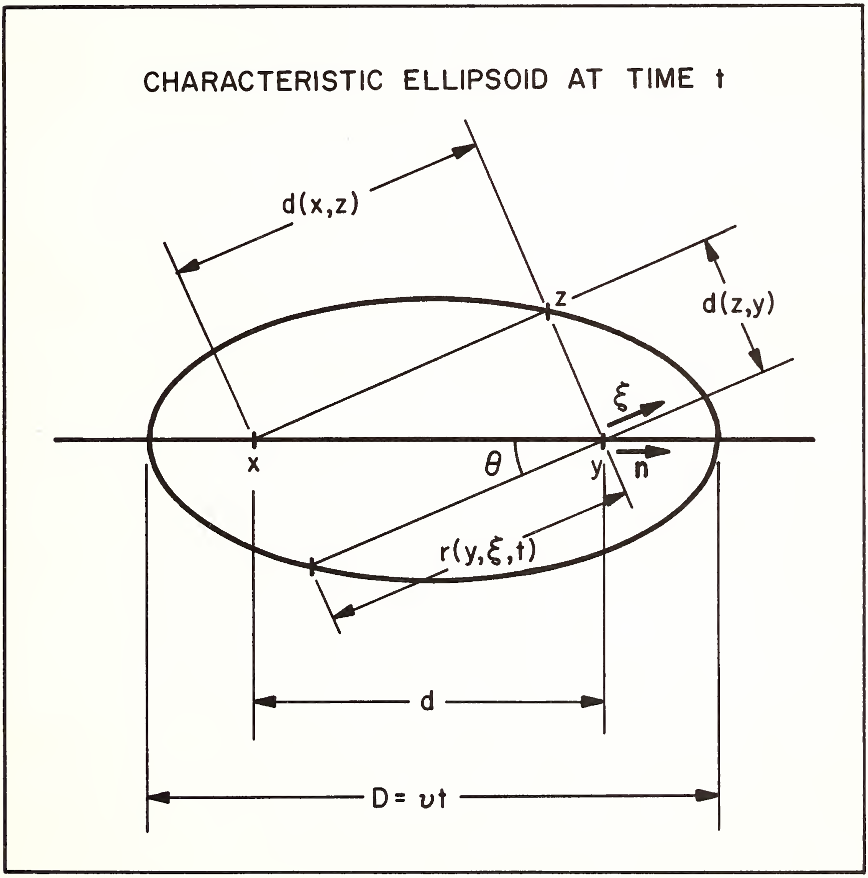

FIG. 5.8 The characteristic ellipsoid relative to the source at x and receiver at y at time t.

For the case at hand these distances are all initially considered in terms of times of travel $t(x,z)$ and $t(z,y)$ across the respective distances and we are interested in all those points z in X such that:

$$d(x,z) + d(z,y) = vt \tag{1}$$

This defines at each instant $t \geq r/v$ an ellipsoid of revolution in X, with foci x and y. From (1) we see that the major axis of the ellipsoid is of length vt. We call the ellipsoid so defined, the *characteristic ellipsoid* $\mathcal{E}(x,y;t)$ associated with x and y at time $t \geq r/v$. A useful polar representation of $\mathcal{E}(x,y;t)$ with y as pole, is given by the equation:

$$r(y,\xi,t) = \frac{D^2 - d^2}{2(D-d\cos\theta)} \tag{2}$$

where θ is the angle between the unit vectors ξ and $\mathbf{n}$, as in Fig. 5.8, and where we have written:

"D" for vt

"d" for d(x,y)

The eccentricity ε of the characteristic ellipsoid $\mathcal{E}(x,y;t)$ turns out to be d/D. At time t such that $t = d(x,y)/v = r/v$, we have $\varepsilon = 1$. As time increases indefinitely, ε decreases to zero, so that--if the space is infinite in all directions about y--the characteristic ellipsoid approaches a sphere which takes on very nearly the polar form:

$$r(y,\xi,t) \cong \frac{D}{2} = \frac{vt}{2}$$

The exact spherical form of $\mathcal{E}(x,y;t)$ occurs at finite times if $x = y$, i.e., whenever $d = 0$. In such a case, $\mathcal{E}(x,x;t)$ becomes the *characteristic spheroid* $S(x;t)$ with radius $vt/2$.

Time-Dependent R and T Operators and the Natural Solution

With the necessary geometrical preliminaries out of the way we can now adapt the **R** and **T** operators of Sec. 5.1 to the time-dependent case. We shall limit the present discussion to a homogeneous steady medium X with point source at a fixed point 0 and such that the characteristic ellipsoid $\mathcal{E}(0,x;t)$ is contained in X for all t under discussion. We shall then write:

$$\text{"}\mathbf{R}\text{" for } \int_{\Xi} [\ ]\sigma(x;\xi';\boldsymbol{\xi})\, d\Omega(\xi')$$

and:

$$\text{"}\mathbf{T}\text{"} \quad \text{for} \int_0^{r(x,\xi,t)} [\ ]T_{r-r'}(x',\xi)\,dr' \tag{4}$$

Comparing this pair of operators with their namesakes in Sec. 5.1, we see that the essential difference between the two pairs rests in the limit of integration for **T**. Now we can limit the integration to the characteristic ellipsoid $\mathcal{E}(0,s;t)$, whereas before (see Fig. 5.1) the limit of integration for **T** was generally the distance from x to the boundary of X in the direction $-\xi$.

If we go on to write:

$$\text{"}\mathbf{S}^1\text{"} \quad \text{for} \quad \mathbf{RT}$$

and then: (5)

$$\text{"}\mathbf{N}^{n+1}\text{"} \quad \text{for} \quad \mathbf{N}^n\,\mathbf{S}^1$$

for every $n \geq 0$, it follows that we can construct the time-dependent natural solution for the time-dependent equation of transfer (4) of Sec. 3.15, just as in 5.4. In particular the solution verification may be repeated line for line and culminating as in (4) of Sec. 5.4, with the form:

$$\boxed{N(x,\xi,t) = N^o(x,\xi,t) + N^*(x,\xi,t)} \tag{5a}$$

but now each term has a time-dependent interpretation.

Truncated Natural Solution

Just as in the steady case in Sec. 5.5 we may now truncate the time-dependent natural solution and obtain an estimate of the accuracy of the truncated solution. It turns out that the truncation estimates of the time-dependent solution can be much sharper than their steady state counterparts, owing to the use of the characteristic ellipsoid in the time-dependent computations. In this discussion suppose the source starts at $t = 0$ and emits in an arbitrary manner thereafter. The light field sweeps out from 0 as center in the form of a spherical field, building up radiant flux of all scattering orders within the sphere as time goes on.

Let $\overline{N}^o$ be the maximum (or supremum, if need be) of the initial radiance function N^o over the sphere of radius vt, center 0. See Fig. 5.9. Then observe that:

$$N^o\mathbf{S}^1(x,\xi,t) \leq \overline{N}^o\ \rho(1-e^{-\alpha r(\max)}) \tag{6}$$

for every ξ in Ξ at x and time t, where $\rho = s/\alpha$ and where we have written:

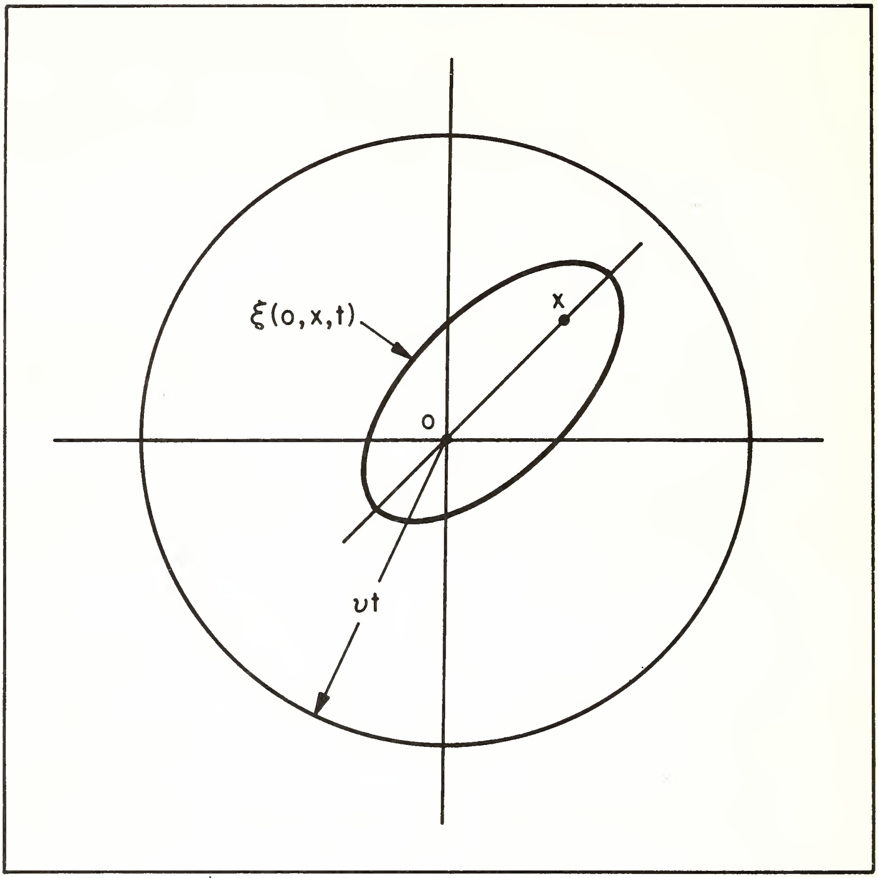

FIG. 5.9 The spherical wave front of the pulse has radius vt. The characteristic ellipsoid relative to 0 and x at time t defines those points of the medium which can send flux to x from 0 at time t.

$$\text{"}r(\max)\text{"} \quad \text{for} \quad \max_{\xi \,\varepsilon\, \Xi} r(x,\xi,t)$$

Hence:

$$r(\max) = (D + d)/2 \quad , \quad D = vt$$

By letting x vary over the spherical region of radius vt, center 0, (6) leads to:

$$N^1(x,\xi,t) = N^o s^1(x,\xi,t) \leq \bar{N}^o \rho(1-e^{-\alpha vt}) \quad , \qquad (7)$$

for every x in X and ξ in Ξ. This may be compared with (3) of Sec. 5.5. Using (7) we can estimate the upper bound of primary scalar irradiance and radiant energy over X in terms of that of residual scalar irradiance or radiant energy. Using the basic idea contained in (7), we can construct a chain of inequalities for n-ary radiances. For (7) yields an upper bound of primary radiance over the sphere of radius vt, center 0, and this upper bound now can be turned around to play the role of $\bar{N}^o$ in the estimate of the next scattering order, namely, $N^2(x,\xi,t)$. Thus in general, since:

$$N^n = N^{n-1}S^1$$

it readily follows that:

$$\boxed{N^n(x,\xi,t) \leq \bar{N}^o[\rho(1-e^{-\alpha vt})]^n} \tag{8}$$

for every x in X, ξ in Ξ, and integer $n \geq 0$. This inequality reduces to (5) of Sec. 5.5 in the steady state, i.e., when $t \to \infty$. The inequality (8) shows that for x sufficiently close to 0 and for small times t,

$$N^n(x,\xi,t) \cong (svt)^n\bar{N}^o \tag{9}$$

where s is the total volume scattering function.

Now, just as in the steady state case of Sec. 5.5, we can estimate the error of truncation of the natural solution series. Thus using (8), we have:

$$N(x,\xi,t) - N^{(k)}(x,\xi,t) = \sum_{j=k+1}^{\infty} [\rho(1-e^{-\alpha vt})]^j$$

$$\leq \bar{N}^o \sum_{j=k+1}^{\infty} [\rho(1-e^{-\alpha vt})]^j$$

Hence:

$$\boxed{N(x,\xi,t) - N^{(k)}(x,\xi,t) \leq \bar{N}^o \frac{[\rho(1-e^{-\alpha vt})]^{k+1}}{1 - [\rho(1-e^{-\alpha vt})]}} \tag{10}$$

for every x in X, and ξ in Ξ at time t. For large times, (10) reduces to (6) of Sec. 5.5. The space and source conditions giving rise to this estimate are stated at the outset of this discussion.

It should now be a relatively simple matter to reduce the preceding analysis to pulselike sources at 0, such as that considered in Sec. 5.6. The general method of analysis and its results developed between (6) and (10), of course remain the same for such sources, but sharper time-dependent estimates of $\bar{N}^o$ are now possible. These truncation estimates are evidently capable of a large variety of treatments and

with the general mode of analysis now clear, each special case is best left to individual treatment by the interested investigator.

5.8 Transport Equations for Residual, Directly Observable, and n-ary Radiant Energy

In this section we shall prepare the way for the extension of the concept of the natural solution of the equation of transfer to the radiant energy field in an optical medium. We shall derive from the time-dependent equations of transfer for the n-ary radiances the corresponding time-dependent transport equations for n-ary radiant energy. We shall eventually find that the latter equations are completely solvable in terms of simple closed algebraic forms in all homogeneous optical media. This fact will allow an important insight into the structure of the associated time-dependent radiance field in the same medium, and thereby shed further light on the difficult optical ringing problem in natural optical media, introduced in Secs. 5.6 and 5.7. We begin with a discussion and solution of the transport equation for zero-order radiant energy (or alternatively, the residual radiant energy) in an optical medium with an arbitrary source. Then the transport equations for nth order radiant energy will be derived along with the transport equations for directly observable radiant energy. Throughout this section the optical medium will be homogeneous with arbitrary sources of radiant flux distributed throughout. The volume scattering function is to be arbitrary but of fixed directional dependence, and unless otherwise specified the scattering-attenuation ratio ρ is also arbitrary but fixed, with $0 < \rho < 1$.

Residual Radiant Energy

In order to help fix the main ideas in the present discussion, let the optical medium X under consideration be depicted as in Fig. 5.10, that is, as an extensive region X with a boundary Y on each point y of which is incident a radiance distribution $N_0(y,\cdot)$ which may be extended into X to obtain initial radiance distributions $N^0(x,\cdot)$ at each point x in X, after the manner of (1) of Sec. 5.1. In the terminology of Sec. 3.10 (see, e.g., (4) of Sec. 3.10) $N^0(x,\xi)$ is the transmitted (or residual) radiance at x in the direction ξ. The alternative term "residual radiance" will be particularly appropriate in the context of the present discussion, and so is singled out for special use.

Suppose now that sources of radiant flux are present within X. This is a relatively new condition since (except for the brief discussion of example 3 of Sec. 3.9), no systematic explicit use of internal sources was required. We have now arrived at a point in our developments where the advent of the special radiometric concept needed for the description of internal sources takes place naturally. We therefore hypothesize the existence of an *emission radiance function* N_η, defined for each time t in some time period and at each point x in X, and direction ξ in Ξ. The dimensions of N_η are precisely those of N_* (radiance per unit length)

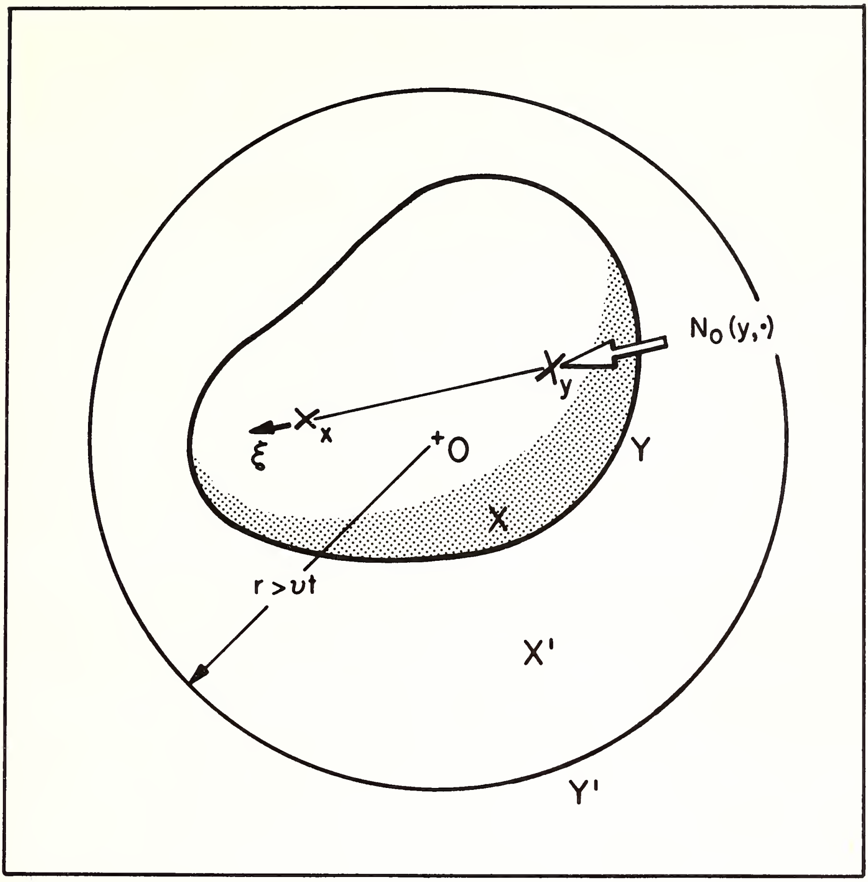

FIG. 5.10 Computing residual radiant energy in medium X.

and the use of N_η may be best understood by keeping this equality of dimensions in mind. Physically, $N_\eta(x,\xi,t)$ is intended to describe the radiance emitted at x and time t per unit length in the direction ξ. We envision $N_\eta(x,\xi,t)$ to be generated by some radiant emission mechanism in X. This mechanism generally takes two distinct forms, which may be in operation singly or simultaneously. These forms are described in Sec. 19 of Ref. [251] and therefore need not be repeated at length here. It suffices for our present purposes to observe that the radiance $N_\eta(x,\xi,t)$ arises generally either through scattering by change in frequency from an arbitrary frequency to the one under consideration, or through the emission processes of conversion of nonradiant energy to radiant energy.

When internal sources, characterized by means of an emission radiance function N_η, are present throughout a medium X, the initial radiance function N^0 is defined throughout X as follows. We write:

$$"N^o(z,\xi,t)" \quad \text{for} \quad N_o(x,\xi,t-r/v)T_r(x,\xi) + \int_0^r N_\eta(x',\xi,t')T_{r-r'}(x',\xi)\,dr' \qquad (1)$$

This definition takes place in the same general geometrical setting of (2) of Sec. 3.10 and reduces to (2) of Sec. 3.10 when X is source-free and the light field is in the steady state. Here as usual $z = x + \xi r$, and $t' = t-r'/v$. A slightly more general definition can be written if X itself has changing inherent optical properties. Also, if scattering with change of frequency is to be explicitly taken into account, we may replace N_η by the true emission function N_e. The details of this more general definition of N^o may be found in Sec. 22 of Ref. [251]. Such generality will not be required in any of our discussions, and so in the interests of simplicity of exposition, the present definition will be retained. Immediately forthcoming from (1) is the equation of transfer for initial radiance in the presence of internal sources:

$$\boxed{\frac{1}{v}\frac{\partial N^o}{\partial t} + \xi \cdot \nabla N^o = -\alpha N^o + N_\eta} \qquad (2)$$

This is obtained by taking the lagrangian derivative of the definitional identity which (1) implies. That is, while following in imagination a photon packet along a natural path through X, we differentiate the right side of (1), by adapting the general procedure used to obtain equation (3) of Sec. 3.15 from equation (1) of that section. Now, we use D/Dt instead of d/dr, where D/Dt is defined in (5) of Sec. 3.15. Equation (2) is a direct generalization of (2) of Sec. 5.2.

We are now ready to define the notion of residual radiant energy and to establish its various analytical representations. By setting n = 0 in the definitions (16) and (17) of Sec. 5.1 we obtain the definitional identity:

$$U^o(X,t) = \frac{1}{v}\int_X \left[\int_\Xi N^o(x,\xi,t)\,d\Omega(\xi)\right] dV(X) \qquad (3)$$

$U^o(X,t)$ is the *residual (or reduced or unattenuated) radiant energy* in X at time t. When X is understood and fixed throughout a discussion (as in the present one) its name may be dropped from the notation and we will write "$U^o(t)$" for the residual radiant energy. The term "residual" is particularly well adapted to the photon interpretation of light. For in that interpretation, $U^o(t)$ is simply the radiant

energy content of X at time t associated with photons which have not been scattered or absorbed relative to the incident and emission sources of flux on X. Thus the photons making up $U^o(t)$ are those left over and in their original unscattered state after t units of time have elapsed since the external sources over X (represented by N_o) and the internal sources over X (represented by N_η) have been turned on.

Transport Equation for Residual Radiant Energy

The transport equation for residual radiant energy can be obtained directly from (2) by applying the integral operations occurring in (3) to each side of (2). Thus, integrating (2) term by term, the time derivative term becomes:

$$\frac{1}{v}\frac{\partial}{\partial t}\int_X \left[\int_\Xi N^o(x,\xi,t)\, d\Omega(\xi)\right] dV(x) = \frac{\partial U^o(t)}{\partial t} \tag{4}$$

Next, we observe that the spatial derivative term may be written as:

$$\nabla \cdot (\xi N^o) \quad ,$$

since ξ is a variable independent of location on X. Then we observe that the integral:

$$\int_\Xi \xi\, N^o(x,\xi,t)\, d\Omega(\xi)$$

defines the residual radiance counterpart to the vector irradiance function **H**, as developed in Sec. 2.8. If we write "$\mathbf{H}^o(x,t)$" for the preceding integral, we can then go on to perform the remaining integration, as required by (3), to obtain:

$$\int_X \nabla \cdot \mathbf{H}^o(x,t)\, dV(x)$$

which by the divergence theorem may be written as a surface integral of **H** over the boundary Y of X; thus:

$$\int_X \nabla \cdot \mathbf{H}^o(x,t)\, dV(x) = -\int_Y \mathbf{H}^o(x,t) \cdot \mathbf{n}(x)\, dA(x) \quad , \tag{5}$$

where $\mathbf{n}(x)$ is the unit inward normal to X at each x on Y, and A is the area measure of Y. Suppose we write:

$$"\overline{P}^{o}(t)" \quad \text{or} \quad "\overline{P}^{o}(Y,t)" \quad \text{for} \quad \int_{Y} \mathbf{H}^{o}(x,t) \cdot \mathbf{n}(x)\, dV(x) \tag{6}$$

Thus $\overline{P}^{o}(Y,t)$ is the net inward flux to X across the boundary Y of X. Finally we write:

$$"P_{\eta}(t)" \quad \text{or} \quad "P_{\eta}(X,t)" \quad \text{for} \quad \int_{X} \left[\int_{\Xi} N_{\eta}(x,\xi,t)\, d\Omega(\xi) \right] dV(x) \tag{7}$$

Thus $P_{\eta}(X,t)$ is the input radiant flux over X at time t. Assembling the results summarized in (4)-(7), equation (2) becomes:

$$\boxed{\frac{dU^{o}(t)}{dt} = -\frac{U^{o}(t)}{T_{\alpha}} + \overline{P}^{o}(t) + P_{\eta}(t)} \tag{8}$$

where we have written:

$$"T_{\alpha}" \quad \text{for} \quad \frac{1}{v\alpha} \tag{9}$$

Equation (8) is the requisite *transport equation for residual radiant energy* in medium X at time t.

The Attenuation Time Constant

The quantity T defined in (9) and which has the dimension of time, is the *attenuation time constant* for X. The significance of T_{α} will become apparent as the discussions of this section proceed. However, a preliminary insight into its significance can be obtained as follows. Imagine all of E_3 to be an infinite homogeneous three-dimensional optical medium about the origin 0. Let the initial radiant energy content of E_3 be zero. Let the sources in E_3 be confined to a point source at 0 which is turned on at time t = 0 and which pours radiant flux out into X at a constant rate P_{η} (i.e., $P_{\eta}(t)$ is independent of t, $t \geq 0$). At any finite time $t > 0$ the spherical wave front traveling outward from 0 is of radius vt. For every $t > 0$, let Y' be any given sphere of radius $r(>vt)$, and let X' be the medium bounded by Y', as in Fig. 5.10.

Under these conditions we have in particular $\overline{P}^{o}(t) = 0$ for every t, $0 \leq t \leq r/v$, and (8) reduces to:

$$\frac{dU^o(t)}{dt} = -\frac{U^o(t)}{T_\alpha} + P_\eta \tag{10}$$

with initial condition:

$$U^o(0) = 0 \quad . \tag{11}$$

The solution of (10), subject to (11), is:

$$\boxed{U^o(t) = U^o(\infty)(1-e^{-t/T\alpha})} \tag{12}$$

over the time interval $(0, r/v)$, and where we have written:

$$"U^o(\infty)" \quad \text{for} \quad P_\eta T_\alpha$$

The significance of T_α now springs into view if we recall a well-known result of elementary circuit analysis concerning the charging of a simple capacitance-resistance DC circuit such as that depicted in Fig. 5.11. When switch S is closed at time $t = 0$, battery B of voltage V pumps electrons along the circuit A which has resistance R, until the capacitor of capacitance C (initially discharged) is fully charged. The amount $q(t)$ of charge on the capacitor

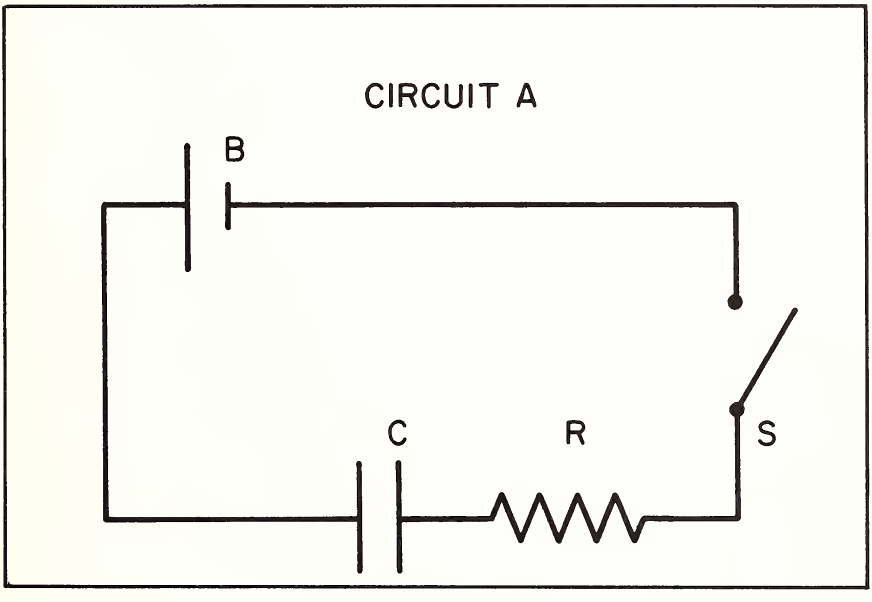

FIG. 5.11 The analogy between an electric circuit and an optical medium.

at time $t \geq 0$ is given by the equation:

$$q(t) = q(\infty)\ (1-e^{-t/RC}) \tag{14}$$

where we have written:

$$\text{"}q(\infty)\text{"} \quad \text{for} \quad CV \tag{15}$$

With the strong structural resemblance between (12) and (14) in mind, we can make the following pairings between the radiative transfer concepts and the electrical circuit concepts:

In the Optical Medium	In the Electrical Circuit
The medium X	The circuit A
The Source Point 0	The battery B
P_η	V/R
$U^o(t)$	q(t)
$1/v$	C
$1/\alpha$	R
T_α (attenuation time constant)	RC (circuit time constant)

Hence the buildup of residual radiant energy in an extensive homogeneous medium X is analogous to the charging of a capacitor in a simple DC capacitor-resistance circuit. The internal source of radiant flux P_η is analogous to the basic current associated with the battery voltage V and circuit resistance R. The capacitance of the circuit is, for given geometry, dependent on the materials of the plates. Thus the smaller the speed of propagation in the material, the larger the capacitance, and the larger the steady state charge $q(\infty)$. Analogously, the smaller the speed of propagation v in the optical medium, all other things being equal, the larger the steady state stored energy $U^o(\infty)$. On this basis (which is not, however, logically compelling) we pair $1/v$ with C. Furthermore, the less dense the conducting material of the circuit, the smaller is the conductance 1/R; similarly, the less dense the material of the optical medium the smaller is α. On this basis we pair $1/\alpha$ with R. The standard circuit time constant RC then pairs off with T_α. This pairing of time constants is relatively strongly suggested by direct comparison of (12) and (14), whereas the suggested pairings of $1/v$ with C and $1/\alpha$ with R are not as strong and, indeed, the pairings may be switched without affecting the important pairing of T_α with RC, the pairing of principal interest at the moment. However, the indicated optical counterparts to R and C are quite interesting to contemplate, particularly when it appears that the analogy between the medium X and the circuit A can be extended quite far by establishing a link with the analogies summarized in the closing paragraph of Sec. 5.6. Apparently, if these analogies can be extended far enough,

then with sufficient care and ingenuity, some of the time-dependent radiative transfer problems can possibly be solved by electrical (or even acoustical) analog methods in which the time-dependent electrical (or reverberating acoustical) field replaces the radiant field.

Just as in the electrical case, the attenuation time constant T_α is the time required for the residual radiant energy to attain 63 percent of its steady state value. Below is given a table for the values of $U^o(t)/U^o(\infty)$ for various values of t in terms of multiples of T_α

TABLE 1

Values of $U^o(t)/U^o(\infty)$ for various values of t in terms of multiples of T_α

$t = nT_\alpha$	$U^o(t)/U^o(\infty)$
T_α	0.63
$2T_\alpha$	0.86
$3T_\alpha$	0.95
$4T_\alpha$	0.98
$5T_\alpha$	0.99

General Representation of Residual Radiant Energy

The solution (12) of the differential equation for residual radiant energy is a special case of the more general solution:

$$\boxed{U^o(t) = U^o(0)e^{-t/T_\alpha} + \int_0^t e^{(t'-t)/T_\alpha} P_o(t')\, dt'} \tag{16}$$

where we have written:

$$\text{"}P_o(t)\text{" for } \overline{P}^o(t) + P_\eta(t) \tag{17}$$

The solution (15) represents the residual radiant energy in a general homogeneous optical medium X with known combined internal and external source flux function P_o, as given by (17).

Transport Equation for n-ary Radiant Energy

We derive next the transport equation for the second main radiometric concept of this section, the n-ary radiant energy $U^m(t)$. The definition of $U^n(t)$ was given in steady state form in Sec. 5.1. Thus we have for every nonnegative integer n,

$$U^n(X,t) = \frac{1}{v} \int_X \left[\int_\Xi N^n(x,\xi,t)\ d\Omega(\xi) \right] dV(x) \tag{18}$$

We shall write "$U^n(t)$" for $U^n(X,t)$ whenever X is understood.

Starting with the time-dependent radiance field in X we apply to (5) of Sec. 5.7 the lagrangian derivative operator D/Dt in exactly the way d/dr was applied to (11) of Sec. 5.1 to yield (1) of Sec. 5.2. We have, as a consequence, for every integer n, $n \geq 1$:

$$\boxed{\frac{1}{v}\frac{\partial N^n}{\partial t} + \xi \cdot \nabla N^n = -\alpha N^n + N_*^n} \tag{19}$$

which is the time-dependent equation of transfer for n-ary radiance N^n, and which is to be compared to (2) above and (1) of Sec. 5.2. Applying the integral operations in (18) to each member of each side of (19), we find that:

$$\frac{1}{v} \int_X \left[\int_\Xi \frac{\partial N^n}{\partial t}\ d\Omega(\xi) \right] dV(x) = \frac{\partial U^n(t)}{\partial t} \ . \tag{20}$$

We write:

$$\text{"}\mathbf{H}^n(x,t)\text{" for } \int_\Xi \xi N^n(x,\xi,t)\ d\Omega(\xi) \tag{21}$$

and

$$\text{"}\bar{P}^n(t)\text{" or "}\bar{P}^n(Y,t)\text{" for } \int_Y \mathbf{H}^n(x,t) \cdot \mathbf{n}(x)\, dA(x) \tag{22}$$

wnere $\mathbf{n}(x)$ is defined as in (6). Finally we observe that:

$$\int_X \left[\int_\Xi N_*^n(x,\xi,t)\ d\Omega(\xi) \right] dV(x) = s\ U^{n-1}(t) \tag{23}$$

With the results (20) through (23) in mind, (19) yields up the following *transport equation for n-ary radiant energy:*

$$\boxed{\frac{dU^n(t)}{dt} = -\frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s} + \bar{P}^n(t)} \tag{24}$$

for every integer $n \geq 1$. The main details of derivation of (24) thus proceed as in the case of the residual radiant energy (8). Here we have written:

$$"T_s" \quad \text{for} \quad 1/vs \tag{25}$$

In equation (24), $\bar{P}^n(t)$ is the *net inward radiant flux* across the boundary Y of X at time t. The radiant flux $\bar{P}^n(t)$ has scattering order n relative to that of $\bar{P}^o(t)$. A term by term interpretation of (24) is instructive: the time rate of change of n-ary radiant energy in X at time t is the sum of a growth term $U^{n-1}(t)/T_s$ (which is the rate of conversion of (n-1)-ary scattered energy into n-ary scattered energy), a decay term $-U^n(t)/T_\alpha$ (which is the rate of conversion of n-ary energy into (n+1)ary energy and nonradiant energy), and finally a general *net rate* of growth term giving the net balance of influx and efflux of n-ary radiant energy across the boundary of X. The quantity T_s is the *scattering time constant* for the medium X. It is a concept which helps write (24) in a uniform manner in terms of the fundamental timelike quantities T_α and T_s.

Transport Equation for Directly Observable Radiant Energy

The radiant energy U associated with directly observable radiance N, using a standard radiance meter is called the *directly observable* radiant energy. This energy is to be held both in conceptual and empirical contrast to the n-ary radiant energy U^n, $n \geq 1$, which is not directly observable in practice. (The residual radiant energy is indirectly observable using techniques alluded to in Sec. 3.10 and Sec. 16 of Ref. [251].) We now derive the transport equation for U(t). We begin with the definitional identity:

$$U(x,t) = \frac{1}{v} \int_X \left[\int_\Xi N(x,\xi,t)\ d\Omega(\xi) \right] dV(x) \tag{26}$$

based on (2) and (12) of Sec. 2.7. As usual we shall drop reference to X, when X is understood.

Starting with the time-dependent radiance equation (4) of Sec. 3.15, we now apply the integral operations in (26) to each side of the transfer equation and obtain, in a manner analogous to that culminating in (8) and (24) above, the result:

$$\boxed{\frac{dU(t)}{dt} = -\frac{U(t)}{T_a} + \overline{P}(t) + P_\eta(t)} \tag{27}$$

This is the *transport equation for directly observable radiant energy.* In the equation we have written:

$$"T_a" \quad \text{for} \quad \frac{1}{va}$$

and where a in turn is the value of the constant volume absorption function in X. Furthermore, we have written:

$$"\overline{P}(t)" \text{ or } "\overline{P}(Y,t)" \quad \text{for} \quad \int_Y \mathbf{H}(x,t) \cdot \mathbf{n}(x)\, dA(x) \tag{28}$$

The unit vector $\mathbf{n}(x)$ is defined as in Fig. 5.10, and so $\overline{P}(t)$ is the net inward radiant flux into X over the boundary Y of X.

The Natural Solution for Directly Observable Radiant Energy

It is a relatively easy matter to verify (using (5a) of Sec. 5.7) that:

$$U(X,t) = \sum_{j=0}^{\infty} U^j(X,t) \tag{29}$$

holds for every $t \geq 0$, where $U(x,t)$ is defined as in (26) and the $U^j(X,t)$ are defined as in (18). Thus, once each $U^j(X,t)$, $j \geq 0$, is known, $U(X,t)$ is known and computable. Equation (29) represents the natural solution of the directly observable radiant energy.

In the case of radiant energy the natural solution procedure is not as vitally essential in the solution of $U(t)$ as in the natural solution procedure for the case of radiance in Secs. 5.6 and 5.7. Indeed, the solution of (27) is written down quite readily, assuming $\overline{P}(t)$ and $P_\eta(t)$ given. Thus, writing,

$$"P(t)" \quad \text{for} \quad \overline{P}(t) + P_\eta(t) \tag{30}$$

we have, analogously to (16):

$$\boxed{U(t) = U(0)e^{-t/T_a} + \int_0^t e^{(t'-t)/T_a}\, P(t')\, dt'} \tag{31}$$

The quantity T_a is the *absorption time constant* for X and is related to T_α and T_s as follows:

$$\frac{1}{T_\alpha} = \frac{1}{T_a} + \frac{1}{T_s} \tag{32}$$

The natural solution procedure for radiant energy is, however, quite useful in throwing light on the inner workings of time-dependent light fields, for the solutions of the transport equations for U^n are readily obtained in simple closed forms which are quite amenable to all manners of explicit, rearrangements and manipulations. Some of the properties of time dependent radiant energy fields will be explored in the next few sections.

We conclude this section with an important observation which will facilitate the studies below. This concerns the connection between the net fluxes $\overline{P}^n(t)$, $n \geq 0$ occurring in (8) and (24), and the net flux $\overline{P}(t)$ occurring in (27). This connection is established by means of the natural solution representation of the directly observable radiant energy U(t) as given in (29). Thus, by summing over all $n \geq 1$ in (24):

$$\frac{d}{dt}\sum_{n=1}^{\infty} U^n(t) = -\frac{1}{T_\alpha}\sum_{n=1}^{\infty} U^n(t) + \frac{1}{T_s}\sum_{n=1}^{\infty} U^{n-1}(t) + \sum_{n=1}^{\infty} \overline{P}^n(t)$$

and adding to these terms the corresponding terms of (8), we obtain::

$$\frac{dU(t)}{dt} = -\frac{U(t)}{T_a} + \sum_{n=0}^{\infty} \overline{P}^n(t) + P_\eta(t)$$

comparing this with (27) we conclude that:

$$\overline{P}(t) = \sum_{n=0}^{\infty} \overline{P}^n(t) \quad . \tag{33}$$

5.9 Solutions of the n-ary Radiant Energy Equations

We shall now solve the transport equation for n-ary scattered radiant energy for every $n \geq 1$, and deduce from the solutions several interesting properties of the scattering order decomposition of natural light fields. These properties are both of intrinsic interest and of use in furthering the natural solution of the radiance field in optical

media. They are also helpful in studying the light storage problems in such media. These latter two applications will be considered in Secs. 5.12 and 5.13. For the present we concentrate on the immediate mathematical and physical features of the transport equations for U^n. Throughout this section, unless specifically noted otherwise, the optical medium will be as in Sec. 5.8, that is homogeneous, with arbitrary sources, arbitrary directional structure for σ, and arbitrary fixed ρ, $0 < \rho < 1$.

Natural Integral Representations of n-ary Radiant Energy

Starting with (24) of Sec. 5.8, we treat the indicated differential equation, for given $n \geq 1$, as an ordinary differential equation with unknown function U^n, and known functions U^{n-1} and $\bar{P}^n$, and with given parameters T_α, T_s. The initial condition for U^n is:

$$U^n(0) = 0 \quad , \tag{1}$$

for every $n \geq 0$. The general solution under this condition can therefore be patterned after (16) or (31) of Sec. 5.8 with the initial values set to zero. Specifically:

$$U^n(t) = \int_0^t e^{(t'-t)/T_\alpha} \left[\frac{U^{n-1}(t')}{T_s} + \bar{P}^n(t') \right] dt' \tag{2}$$

Now, to simplify matters we shall assume that:

$$\bar{P}^n(t) = 0 \tag{3}$$

for every $n \geq 0$ over a given interval $(0, t_1)$ of time which is to include the time interval in which we shall be interested in the solutions of (24) of Sec. 5.8. Physically this means in effect that the collective expanding wave fronts of all sources in X are completely within the boundary Y of X over the time interval $(0, t_1)$. See Figure 5.10. With assumption (3) in force, (2) becomes:

$$U^n(t) = \frac{1}{T_s} \int_0^t e^{(t'-t)/T_\alpha}\, U^{n-1}(t')\, dt'$$

$$= \frac{e^{-t/T_\alpha}}{T_s} \int_0^t e^{t'/T_\alpha}\, U^{n-1}(t')\, dt' \tag{4}$$

which holds for $n \geq 1$ and $0 \leq t \leq t_1$. The form of (4) suggests a recursive construction of $U^n(t)$ starting with $n = 1$ and using knowledge of $U^o(t)$ as given in (16) of Sec. 5.8. By (3), $P_o(t)$ in (16) of Sec. 5.8 uses only the internal source function P_η. Hence $U^n(t)$ should be expressible in terms of $U^o(t)$ (or $P_\eta(t)$) along with T_s and T_α. Thus, starting with (4) now applied to $U^{n-1}(t)$, $n-1 \geq 1$, we have:

$$U^n(t) = \frac{1}{T_s}\int_0^t e^{(t'-t)/T_\alpha}\left[\frac{1}{T_s}\int_0^{t'} e^{(t''-t')/T_\alpha}\, U^{n-2}(t'')\, dt''\right] dt'$$

$$= \frac{e^{-t/T_\alpha}}{T_s^2}\int_0^t (t-t')\, e^{t'/T_\alpha}\, U^{n-2}(t')\, dt' \qquad (5)$$

This process can be continued as long as the scattering order in the integrand is greater than zero. The pattern forming in (4) and (5) is clear. Applying (4) once again, now to U^{n-2} the pattern is crystallized:

$$U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^3}\int_0^t \frac{(t-t')^2}{2}\, e^{t'/T_\alpha}\, U^{n-3}(t')\, dt' \quad . \qquad (6)$$

Thus, applying the representation (4) in all k times, $0 \leq k \leq n-1$, we have for $U^n(t)$:

$$U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^{k+1}}\int_0^t \frac{(t-t')^k}{k!}\, e^{t'/T_\alpha}\, U^{n-k-1}(t')\, dt' \qquad (7)$$

If in (7) we let $k = n-1$, then the desired integral representation of $U^n(t)$, $0 \leq t \leq t_1$ is obtained:

$$\boxed{U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^n}\int_0^t \frac{(t-t')^{n-1}}{(n-1)!}\, e^{t'/T_\alpha}\, U^o(t')\, dt'} \qquad (8)$$

or, in terms of P_η:

$$\boxed{U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^n} \int_0^t \frac{(t-t')^n}{n!} e^{t'/T_\alpha} P_\eta(t')\, dt'} \tag{9}$$

Equations (8) or (9) are the desired integral representations of $U^n(t)$. Observe that (8) holds for $n \geq 1$ and (9) holds for $n \geq 0$.

Natural Closed Form Representations of n-ary Radiant Energy

The formulas (8) or (9) are the requisite representations of $U^n(t)$ under the given initial conditions (1), and the conditions on the medium hypothesized in (3) and at the outset of this section. In order to evaluate the integrals we must specify the nature of U^n or P_η over the time interval $(0, t_1)$. We now illustrate the use of (9) by choosing two important instances of P_η. The first instance is where P_η is the Dirac-delta function centered at $t = 0$ and with radiant energy content U_η. The second instance is where P_η is constant valued over $(0, t_1)$ with its constant magnitude denoted by "P_η". In the first instance, we have:

$$U^n(t) = U_\eta \left(\frac{t}{T_s}\right)^n \frac{e^{-t/T_\alpha}}{n!} \tag{10}$$

for

$$P_\eta(t) = U_\eta(t)\ \delta(t)$$

over the interval $(0,t_1)$ and for $n \geq 0$. We shall refer to this case as the *optical reverberation case* (cf. the introduction to Sec. 5.6).

The second instance yields the representation:

$$U^n(t) = \left[\frac{T_\alpha}{T_s}\right]^n U^o(\infty) \left[1 - \left(\sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!}\right) e^{-t/T_\alpha}\right] \tag{11}$$

for

$$P_\eta(t) = P_\eta \quad .$$

over the interval $(0, t_1)$ and for $n \geq 0$. Here $U^o(\infty)$ is as defined in (13) of Sec. 5.8. These two specific instances of (9) are verified by direct integration of (9) in each case.

General Integral Representations of n-ary Radiant Energy

The integral representation (9) of U^n will now be generalized to the case for which the initial conditions on U^j, $j < n$, are arbitrary. That is, we now relax the conditions (1). However, we shall retain condition (3). The resultant representation will permit the construction of relatively general representations of the time-dependent n-ary radiant energy in a homogeneous medium for which the wave fronts of internal sources have not yet passed the boundaries. Thus, by successive applications of the type of solution displayed in (16) of Sec. 5.8, we eventually arrive at:

$$\boxed{U^n(t) = \left[U^n(0) + \left(\frac{t}{T_s}\right) U^{n-1}(o) + \dots + \frac{1}{n!}\left(\frac{t}{T_s}\right)^n U^o(0)\right] e^{-t/T_\alpha} + \frac{e^{-t/T_\alpha}}{T_s^n} \int_0^t \frac{(t-t')^n}{n!} e^{t'/T_\alpha} P_\eta(t')\, dt'} \tag{12}$$

This is the desired generalization of (9), which holds for $n \geq 0$.

Standard Growth and Decay Formulas for n-ary Radiant Energy

Of the infinite variety of possible time-dependent radiant energy fields attainable in principle via (12), two types stand out as particularly interesting. These are sufficiently instructive to isolate and set up here as standards. The first of these light fields is that given by (11) above. This equation we shall call the *standard growth formula* for U^n. Recall that in this case the initial values for the U^j, $j \leq n$, are all zero and that P_η is a positive constant over some time interval $(0, t_1)$. Suppose we write:

$$\text{"}F_n(t/T_\alpha)\text{"} \quad \text{for} \quad e^{-t/T_\alpha} \sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!} \tag{13}$$

Then we summarize the *standard growth formula* as follows: If

(a) The optical medium is homogeneous.

(b) $U^n(0) = 0$ and $P_\eta(t) = P_\eta$ for t in $(0,t_1)$ and $n \geq 0$.

(c) $\overline{P}^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$\boxed{U^n(t) = U^n(\infty)[1-F_n(t/T_\alpha)]} \tag{14}$$

for every t in $(0,t_1)$ and $n \geq 0$.

The second standard case is that which describes the decay of the n-ary light field from a given steady state level. Thus if an opaque curtain were suddenly drawn over the ocean in which previously all internal radiant sources were turned off, the following *standard decay formula* for U^n would describe very closely the decay of $U^n(t)$ for $t \geq 0$ for every $n \geq 0$ in the ocean; thus: If

(a) The optical medium is homogeneous.

(b) $U^n(0) = \rho^n U^o(0)$ and $P_\eta = 0$ for t in $(0,t_1)$ and $n \geq 0$.

(c) $\overline{P}^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$\boxed{U^n(t) = U^n(0)\; F_n(t/T_\alpha)} \tag{15}$$

for every t in $(0,t_1)$ and $n \geq 0$.

A few words about condition (b), the initial condition for U^n, are in order. An examination of the general representation (11) of $U^n(t)$ shows that at steady state (i.e., the limit of $U^n(t)$ as $t \to \infty$) the various magnitudes $U^n(\infty)$ are not arbitrary. Indeed, they generally depend on P_η and the initial values $U^n(0)$, as explicitly shown in (12). Hence when a steady state light field begins to decay after sources have been turned off, the initial values $U^n(0)$, $n \geq 0$ are generally not expected to be independent of each other. For example, if the standard growth conditions are in effect, then (11) shows that:

$$U^n(\infty) = \left[\frac{T_\alpha}{T_s}\right]^n U^o(\infty) = \rho^n U^o(\infty)$$

$$= \rho^n P_\eta T_\alpha$$

for every $n \geq 0$. Thus we see that the standard decay formula is intended to describe the decay of a light field which has been attained under standard growth conditions as given by (14) for $t \to \infty$.

We can combine the standard growth and decay formulas (14) and (15) into a single standard formula as follows: If

(a) The optical medium is homogeneous.

(b) $U^n(0)$, $n \geq 0$ is given as steady state value attained under a previous standard growth condition and $P_\eta(t) = P_\eta$ for t in $(0,t_1)$.

(c) $\bar{P}^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$\boxed{U^n(t) = U^n(\infty) + [U^n(0) - U^n(\infty)]\, F_n(t/T_\alpha)} \tag{16}$$

and where $U^n(\infty)$ is determined by (14) for the present source condition. As an interesting consistency check, observe that if the previous steady state condition (b) above is induced by P_η as given in (b), then $U^n(t)$ in (16) is independent of time, because $U^n(0) = U^n(\infty)$.

As a final standard type of growth and decay formula, we consider the case in which a standard growth begins at $t = 0$ and continues until time t_o, at which time the source is shut off and the existing light field decays from that point on until some arbitrary time t_1 under standard decay conditions. Equation (12) shows that the decay formula is:

$$U^n(t) = \left[U^n(t_o) + \left(\frac{t-t_o}{T_s}\right)U^{n-1}(t_o) + \dots + \left(\frac{t-t_o}{T_s}\right)^n U^o(t_o)\right] e^{-(t-t_o)/T\alpha} \tag{17}$$

for $t_o \leq t \leq t_1$ and $n \geq 0$. For $t \leq t_o$, $U^n(t)$ is given by (11). Formula (17) may be used to describe the transient radiant energy fields induced in large bodies of air or water by radiant sources which are intermediate between the Dirac-delta pulse and the steady source described in (10) and (11). Since any source output P_η over a time interval $(0,t_o)$ can be approximated by a step function, we see that by superimposing fields of the type given by (17), we can represent n-ary radiant energy fields induced by finite non-constant sources under the general conditions of this section.

5.10 Properties of Time-Dependent n-ary Radiant Energy Fields and Related Fields

We now turn to examine in detail some of the more intuitively interesting properties of time-dependent radiant energy fields. In order to present the properties in their simplest forms, we shall adopt for study throughout this section a light field evolving under either *standard growth or decay conditions* or *optical reverberation conditions* in an

optical medium X over a time interval $(0,t_1)$ (Sec. 5.9). It will be clear from the results stated below how analogous or complementary statements and properties can be formulated under still more general conditions. We begin with a study of some of the fine-structure properties of n-ary radiant energy fields and then go on to a formulation of the various representations of related radiant energy quantities.

Some Fine-Structure Properties of n-ary Radiant Energy

Property 1. *Let t be a fixed time in $(0,t_1)$. Then the sequence $U^0(t)$, $U^1(t)$, ..., $U^n(t)$, ... of n-ary radiant energies at time t is a monotonic decreasing sequence with limit 0.* The proof of this property is based on (14) of Sec. 5.9. By (13) of Sec. 5.9 we see that:

$$\lim_n F_n(t/T_\alpha) = 1 \tag{1}$$

Hence by noting that $0 < \rho < 1$, we see that:

$$\lim_n U^n(\infty) = 0$$

so that

$$\lim_n U^n(t) = 0$$

for t in $(0,t_1)$. As for the monotonicity of the sequence, it suffices to note that:

$$\frac{U^{n+1}(t)}{U^n(t)} = \rho \frac{1-F_{n+1}(t/T_\alpha)}{1-F_n(t/T_\alpha)} \tag{2}$$

and that $F_n(t/T_\alpha)$ increases monotonically, with n, to unity. This may be seen by verifying that:

$$0 < 1 - F_{n+1}(t/T_\alpha) < 1 - F_n(t/T_\alpha) < 1$$

for every $n \geq 0$ and every positive t. The limit part of property 1 follows also from (2) by using the ratio test for convergent infinite series.

Property 2. *Under standard growth conditions,*

$$\frac{dU^n(t)}{dt} = P_\eta \left[\frac{t}{T_s}\right]^n \frac{e^{-t/T_\alpha}}{n!} > 0$$

for every t in $(0,t_1)$. The proof is immediate. For example, one may use (14) of Sec. 5.9 directly with the calculus, or one may use algebra with the fact that $dU^n(t)/dt$ is the

difference given in (24) of Sec. 5.8, with $P^n(t) = 0$. Property 2 shows in particular that each n-ary radiant energy component increases monotonically with time. Property 2 is to be compared with:

Property 3. *Under standard decay conditions*

$$\frac{dU^n(t)}{dt} = -\frac{U^n(0)}{T_\alpha}\left[\frac{t}{T_s}\right]^n \frac{e^{-t/T_\alpha}}{n!} < 0$$

for every t in $(0,t_1)$. The proof is immediately obtainable from (15) of Sec. 5.9. Hence the rates of growth and decay of n-ary radiant energy under standard conditions are, to within a constant multiplicative factor, identical in structure within a given space.

Property 4. *Under standard growth conditions,*

$$\frac{U^{n+k}(t)}{U^n(t)} < \rho^k$$

for every t in $(0,t_1)$ and positive integers n, k. This follows from property 2 and (24) of Sec. 5.8 with $P^n(t) = 0$. The inequality is reversed under standard decay conditions.

Property 5. *In the steady state of the standard growth process,*

$$U^n(\infty) = \rho^n U^o(\infty)$$

for every $n \geq 0$. Hence:

$$\frac{U^{n+k}(\infty)}{U^n(\infty)} = \rho^k$$

for every pair n, k of nonnegative integers.

Property 6. *In the optical reverberation case (equation (10) of Sec. 5.9) we have the ratio:*

$$U^n(t)/U^{n-1}(t) = \frac{vts}{n} = t/nT_s$$

for $n \geq 1$ and t in $(0,t_1)$. Thus, the ratio of successive n-ary radiant energy contents increases linearly with increasing time and decreases hyperbolically with increasing n.

Property 7. *In the optical reverberation case with point source (equation (10) of Sec. 5.9) $U^n(t)$, for a given scattering order, attains a maximum when the radius of the wave front is n times the attenuation length $1/\alpha$. Further, for any given total volume scattering value s and time t in $(0,t_1)$, that component $U^n(t)$ is maximal whose order n makes the absolute value of*

$$\left(\frac{vts}{n}\right) - 1 = (t/nT_s) - 1$$

a minimum. The geometric content of properties 6 and 7 are summarized in part (a) of Figure 5.12.

Property 8. *In the optical reverberation case, the directly observable radiant energy U(t) is given by:*

$$U(t) = U_\eta \, e^{-t/T_a}$$

The proof rests on (10) of Sec. 5.9 and (29) of Sec. 5.8 and the simple calculation:

$$U(t) = \sum_{j=0}^{\infty} U^j(t) = U_\eta \, e^{-t/T_\alpha} \sum_{j=0}^{\infty} \frac{\left(\frac{t}{T_s}\right)^j}{j!}$$

$$= U_\eta \, e^{-t/T_\alpha} \cdot e^{t/T_s} = U_\eta \, e^{-t/T_a} \quad ,$$

in which (32) of Sec. 5.8 was used. It follows immediately from property 8 that, in optical media with no absorption, i.e., for which a = 0, U(t) is independent of t in the reverberation case. Part (b) of Figure 5.12 gives plots of $U^n(t)$ for the first four scattering orders in the optical reverberation case in which a = 0 and $U_\eta = 1$. In the figure we have

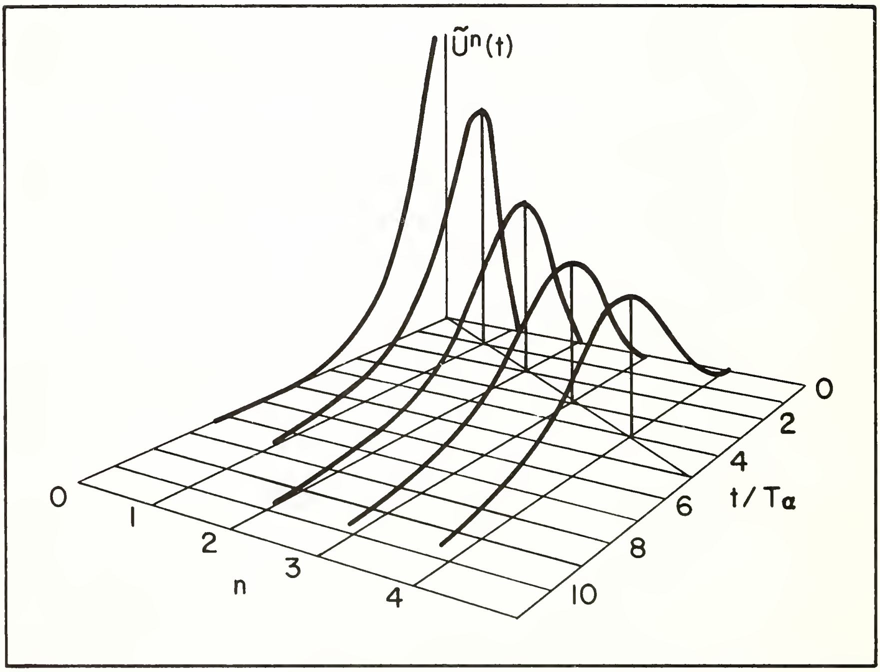

FIG. 5.12(a) The geometric version of property 7 of scattered radiant energy.

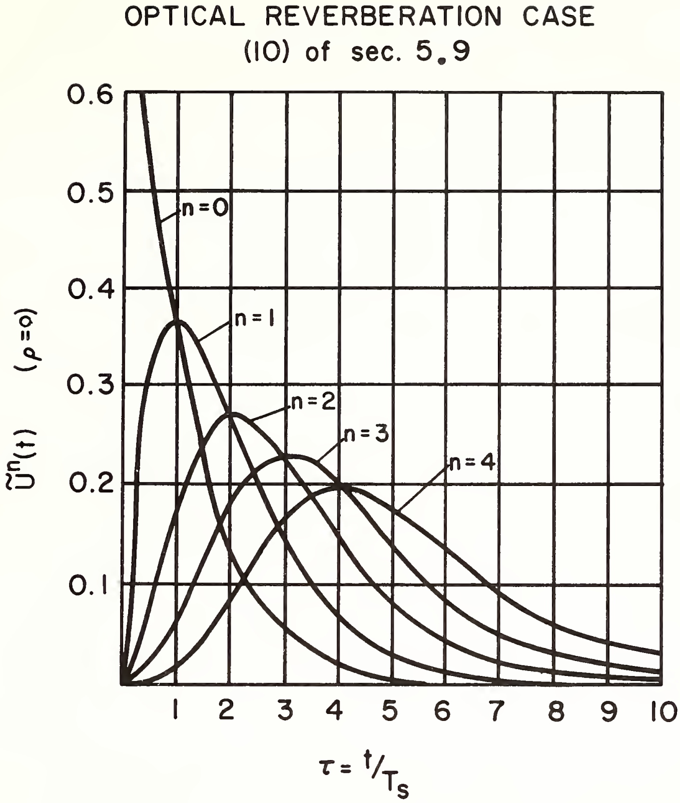

FIG. 5.12(b) The geometric version of property 7 of scattered radiant energy.--Concluded.

written "τ" for t/T_S. Thus the medium is a nonabsorbing medium ($\rho = 0$) with conserved directly observable energy. Note how the scattering order components of U(t) well up one after another, reaching their maxima, as described by property 7. Finally, according to property 8, the sum of the ordinates of all the curves at each τ should add up to unity.

Scattered, Absorbed, and Attenuated Radiant Energies

We now round out the roster of the types of radiant energy fields most commonly encountered in theoretical discussions of time-dependent light fields. Until further notice, source conditions are arbitrary and with $\overline{P}(t) = 0$.

So far we have introduced the residual radiant energy ((3) of Sec. 5.8), the n-ary radiant energy ((19) of Sec. 5.8), and the directly observable radiant energy ((26) of

Sec. 5.8) with its natural representation ((29) of Sec. 5.8). By writing:

$$\text{"}U^*(t)\text{"} \quad \text{for} \quad \sum_{j=1}^{\infty} U^j(t) \tag{3}$$

we define the *scattered* (or diffuse) *radiant energy* (in X) at time t. We then have from (29) of Sec. 5.8 the following radiant energy counterpart to the time-dependent integral equation of transfer (cf. (4) of Sec. 5.4):

$$U(t) = U^o(t) + U^*(t) \tag{4}$$

Using the emission radiant flux function P_η and recalling that we have set $\overline{P}(t) = 0$ for t in $(0, t_1)$, let us write:

$$\text{"}U(t;\alpha)\text{"} \quad \text{for} \quad \int_0^t P_\eta\,(t')\,dt' - U^o(t) \tag{5}$$

for t in $(0, t_1)$. The meaning of this new radiant energy becomes clear when it is recalled that $U^o(t)$ is the residual (i.e., the *unattenuated*) radiant energy. Therefore, since the integral gives the total radiant energy input to the medium, the difference in (5) must be all the energy present at time t that has undergone attenuation (absorption or at least one scattering operation). We call $U(t;\alpha)$ the *attenuated radiant energy* (in the medium X) at time t. Only part of $U(t;\alpha)$ is detectable. In fact, the detectable part of $U(t;\alpha)$ is precisely $U^*(t)$. Thus let us write:

$$\text{"}U(t;a)\text{"} \quad \text{for} \quad U(t;\alpha) - U(t;s) \tag{6}$$

where, for uniformity of notation and heuristic purposes, we have agreed momentarily to write

$$\text{"}U(t;s)\text{"} \quad \text{for} \quad U^*(t) \quad . \tag{7}$$

Then from (6) we have:

$$U(t;\alpha) = U(t;a) + U(t;s) \quad , \tag{8}$$

a formula remarkably similar in structure to the basic relation:

$$\alpha = a + s$$

derived from (4) of Sec. 4.2. We call $U(t;a)$ the *absorbed radiant energy* (in X) at time t. The absorbed radiant energy is radiant energy that has disappeared from the present radiometric scene via absorption processes.

Representations of $U(t;\alpha)$,
$U(t;s)$, and $U(t;a)$

The transport equations for the three auxiliarly radiant energies and their solutions are relatively easy to obtain. We shall illustrate the power of the natural solution procedure by basing the derivations of these equations and representations directly on the knowledge of the n-ary radiant energies developed so far.

We begin with the derivation of the differential equation for attenuated radiant energy $U(t;\alpha)$. From the definition (5) we have

$$\frac{dU(t;\alpha)}{dt} = P_\eta(t) - \frac{dU^o(t)}{dt} \quad .$$

From (8) of Sec. 5.8 we obtain:

$$\boxed{\frac{dU(t;\alpha)}{dt} = \frac{U^o(t)}{T_\alpha}} \tag{9}$$

recalling that the condition $\bar{P}^n(t) = 0$ is in force for every $n \geq 0$ (hence $\bar{P}^o(t) = 0$, in particular, holds). This elegant formula for the growth rate of $U(t;\alpha)$ shows perhaps most clearly the reservoir source of $U(t;\alpha)$ (namely, $U^o(t)$) and the main line which taps the reservoir (namely, T_α, i.e., attenuation). At standard steady state (9) shows that:

$$\frac{dU(\infty;\alpha)}{dt} = P_\eta \tag{10}$$

Thus in the steady state attained under standard growth conditions the rate of increase of $U(t;\alpha)$ is precisely the input rate P_η, so that attenuated radiant energy in the medium increases as fast as it is put into the medium by the source.

Next we consider the scattered radiant energy $U(t;s)$, or "$U^*(t)$" as we would call it ordinarily. The representation (3) of $U(t;s)$ gives rise to the associated differential equation for $U(t;s)$ by computing (with the help of (24) of Sec. 5.8) the following derivative:

$$\begin{aligned}\frac{dU(t;s)}{dt} &= \sum_{j=1}^{\infty} \frac{dU^j(t)}{dt} \\ &= \sum_{j=1}^{\infty}\left(- \frac{U^j(t)}{T_\alpha} + \frac{U^{j-1}(t)}{T_s}\right) \\ &= \left(- \frac{1}{T_\alpha} + \frac{1}{T_s}\right) U(t;s) + \frac{U^o(t)}{T_s} \quad .\end{aligned}$$

Hence:

$$\boxed{\frac{dU(t;s)}{dt} = -\frac{U(t;s)}{T_a} + \frac{U^o(t)}{T_s}} \qquad (11)$$

Here we begin to see some of the utility of the various time constants T_a, T_s, T_α. They serve to remind one of the correct dimensions of each term in an equation or representation, and they serve also to show the physical mechanism associated with that term. Thus we see at a glance from (11) that the rate of growth of U(t;s)--the scattered radiant energy--is augmented by scattering of residual radiant energy $U^o(t)$ and decreased by absorption of scattered radiant energy U(t;s).

There is no need to solve (11) since we need only sum the representations of the $U^j(t)$ in (3) to obtain the desired representation of U(t;s). Thus, under standard growth conditions ((14) of Sec. 5.9):

$$U(t;s) = \sum_{k=1}^{\infty} U^k(t) = \sum_{k=1}^{\infty} U^k(\infty)\ [1-F_k(t/T\alpha)]$$

$$= U^o(\infty) \sum_{k=1}^{\infty} \left[\frac{T_\alpha}{T_s}\right]^k \left[1- \sum_{j=0}^{k} \frac{(t/T_\alpha)^j}{j!}\ e^{-t/T_\alpha}\right]$$

Hence:

$$\boxed{U(t;s) = T_a U^o(\infty)\left[\frac{1}{T_\alpha}(1-e^{-t/T_a}) - \frac{1}{T_a}(1-e^{-t/T_\alpha})\right]} \qquad (12)$$

An alternate representation of U(t;s)is obtained by distributing $T_a U^o(\infty)$ throughout the preceding representation. The result is:

$$U(t;s) = \left[\frac{T_a}{T_\alpha}\right] U^o(\infty)(1-e^{-t/T_a}) - U^o(t) \qquad (13)$$

From this we obtain immediately the representation for the directly observable radiant energy. For, by (4) and (13), we have:

$$\boxed{U(t) = \left[\frac{T_a}{T_\alpha}\right] U^o(\infty)(1-e^{-t/T_a})} \qquad (14)$$

which is clearly a solution of (27) of Sec. 5.8 under standard growth conditions.

Finally the absorbed radiant energy is represented most simply as:

$$\boxed{U(t;a) = P_\eta t - U(t)} \tag{15}$$

under standard growth conditions. This representation follows from (4), (5), and (8). A representation under more general growth conditions is obtained by retaining the integral in (5). The differential equation for U(t;a) under standard growth conditions is readily obtained:

$$\begin{aligned} \frac{dU(t;a)}{dt} &= \frac{dU(t;\alpha)}{dt} - \frac{dU(t;s)}{dt} \\ &= \frac{U^o(t)}{T_\alpha} - \left(- \frac{U(t;s)}{T_a} + \frac{U^o(t)}{T_s}\right) \\ &= \frac{U(t;s) + U^o(t)}{T_a} \end{aligned}$$

Hence:

$$\boxed{\frac{dU(t;a)}{dt} = \frac{U(t)}{T_a}} \tag{16}$$

We have made a point of deriving the differential equation for U(t;a) so as to make possible the comparison between it and (9). The comparison lends valuable insight into the general roles of scattering and absorption in radiative transfer phenomena. Thus, in the case of (16), the reservoir source for U(t;a) is the directly observable radiant energy and the energy is tapped via the process of absorption.

5.11 Dimensionless Forms of n-ary Radiant Energy Fields and Related Fields

We shall now develop the dimensionless forms of the various equations and solutions for n-ary radiant energy, residual radiant energy, directly observable radiant energy, and the related energy fields introduced in Sec. 5.10. We shall also explore the various possibilities for the definition of time constants which are to characterize time-dependent light fields in optical media. Before going on to the details of the discussion, some preliminary observations on physical theories using dimensionless concepts are in order.

When the analytical representation of a natural phenomenon can be placed into such a form that the terms of the new representation are dimensionless, this usually indicates that the given phenomenon is a member of an inclusive class of phenomena whose members exhibit a common mathematical representation, but which ostensibly may have different external appearances. The mathematics used to represent the concepts of electrical network theory is a good example of this kind; for the mathematical procedures employed in that theory are

often equally applicable to problems in mechanical dynamics. As a result of this common understructure, researchers in each of these fields have enriched the mathematical methods of the other by noting the applicability of the same set of techniques in each field of study. (See Sec. 5.15.)

Some of the discussions in this chapter have already indicated that the set of transient radiant energy phenomena may be treated as a member of the class of natural phenomena which includes electrical network behavior ((14) of Sec. 5.8; see also concluding comments of Sec. 5.6). We can also point out that the natural mode of solution leads to radiant energy equations which have the same mathematical structure as the equations governing the growth and decay of families radioactive substances. In this case, the counterparts to n-ary radiant energy U^n are the population counts P_n of the nth species S_n of radioactive atoms which are the decay products of species S_{n-1} and where S_n itself decays into species S_{n+1}. Still other and ostensibly different natural phenomena share the same mathematical substructure as the time-dependent radiant energy equations. For example, interacting biological species S_n often are arranged in a predatory hierarchy so that members of species S_n prey upon those in species S_{n-1} and are in turn preyed upon by those in species S_{n+1}. The time-dependent equations governing the population counts of the nth interacting species--be they animal, vegetable, or mineral--often have a common fundamental mathematical core which is obtainable by stripping away the accidental topography of the equations associated with the particular case. The advantages of attaining such dimensionless formulations lie in the resultant conceptual simplifications and economy of description of natural processes.

The casting into dimensionless form of the basic differential equations of transient radiant energy and their associated solutions has practical as well as conceptual advantages. For example, dimensionless formulas allow the inclusion of a wide range of special cases in a single tabulation or graph, the specific case being recoverable after multiplication by a suitable factor. The dimensionless forms thus compress a huge amount of particular numerical information into a relatively small space.

We turn now to the details of the discussion. For simplicity we shall adopt throughout this section the standard growth conditions in a homogeneous optical medium (re: (14) of Sec. 5.9). The developments of this section may serve as a pattern for generalizations to the nonstandard cases.

Conversion Rules for Dimensionless Quantities

An examination of the various analytic representations of $U^o(t)$, $U^*(t)$, $U(t)$, and related radiant energy concepts in Sec. 5.10, with an eye toward achieving dimensionless versions of these representations, brings to light the essential observation that, without exception, each of the representations within the standard growth context obtains its

dimension of *energy* from the presence of the product $P_n T_\alpha$ in the form of $U^o(\infty)$. For example, (12) of Sec. 5.8 states that

$$U^o(t) = U^o(\infty)(1-e^{-t/T_\alpha})$$

and (11) of Sec. 5.9 states that:

$$U^n(t) = \left[\frac{T_\alpha}{T_s}\right]^n U^o(\infty) \ [1-F_n(t/T_\alpha)]$$

A perusal of $U(t;\alpha)$, $U(t;s)$, (i.e., $U^*(t)$) and $U(t;a)$ in the preceding section will corroborate the observation still further. This leads us to the following definition.

Definition of the Dimensionless form of U. Let "$U^\#$" denote any of the following radiant energy expressions: $U^n(t)$, $U(t;\alpha)$, $U(t;s)$, $U(t;a)$, $U(t)$. Then we shall write:

$$"\tilde{U}^\#" \quad \text{for} \quad U^\#/U^o(\infty)$$

and we call $\tilde{U}^\#$ the *dimensionless* form of U.

The next observation concerns the presence of terms of the form t/T_α, t/T_s, t/T_a, T_α/T_a, T_s/T_a, and T_α/T_s in the various equations constructed so far. These expressions are already dimensionless. The observation to make at present is that these six terms, which involve four separate concepts, can be represented compactly by means of only two distinct concepts, namely the ratio t/T_α and the scattering-attenuation ratio $\rho(=s/\alpha)$. To see this, let us write:

$$"\tau" \quad \text{for} \quad t/T_\alpha \tag{1}$$

We call τ the *relative time.* Its connection with steady state concepts is very close and may be stated succinctly by first writing

$$"L_\alpha" \quad \text{for} \quad 1/\alpha \quad .$$

We call L_α the *attenuation length* associated with the optical medium. Since T_α is $1/v\alpha$, we see that:

$$L_\alpha = vT_\alpha \tag{2}$$

so that:

$$\tau = t/T_\alpha = vt/L_\alpha \tag{3}$$

From (3), τ may be interpreted not only in a temporal sense (i.e., the number of attenuation times in a certain time t), but in a spatial sense, too, namely the number of attenuation lengths in a certain path (traversed by light in real time t). The representation of the six dimensionless terms displayed above may be made in terms of ρ and τ as follows:

TABLE 2

Representation of six dimensionless terms.

t/T_α	τ
t/T_s	$\rho\tau$
t/T_a	$(1-\rho)\tau$
T_α/T_s	ρ
T_α/T_a	$(1-\rho)$
T_s/T_a	$(1-\rho)/\rho$

We are now ready to state the conversion rules by which one is guided to the dimensionless differential equations and associated solutions for the various radiant energy fields. Towards this end, we note that the derivative:

$$\frac{dU\#(t)}{dt}$$

may be written as:

$$\frac{dU\#(\tau)}{d\tau} \cdot \frac{d\tau}{dt} ,$$

where:

$$\frac{d\tau}{dt} = 1/T_\alpha$$

so that:

$$\boxed{\frac{dU\#(\tau)}{d\tau} = T_\alpha \frac{dU\#(t)}{dt}} \qquad (4)$$

Conversion rule 1. *To convert $dU\#(t)/dt$ to dimensionless form under standard growth conditions, multiply by $T_\alpha/U^o(\infty)$ and change all time ratios of the kind t/T_x and T_x/T_y into their equivalent forms in terms of ρ and τ, using Table 2.*

Conversion rule 2. *To convert $U\#(t)$ to dimensionless form under standard growth conditions, multiply by $1/U^o(\infty)$ and change all time ratios of the kind t/T_x and T_x/T_y into their equivalent forms in terms of ρ and τ, using Table 2.*

Dimensionless Forms for $U^o(t)$

Starting with (8) of Sec. 5.8 under the standard growth condition, we have

$$\frac{dU^o(t)}{dt} = - \frac{U^o(t)}{T_\alpha} + P_\eta$$

To apply conversion rules 1 and 2, we write this as:

$$T_\alpha \frac{d\ U^o(t)/U^o(\infty)}{dt} = - U^o(t)/U^o(\infty) + P_\eta T_\alpha/U^o(\infty)$$

and then go on to obtain:

$$\boxed{\frac{d\tilde{U}^o(\tau)}{d\tau} = - \tilde{U}^o(\tau) + 1} \qquad (5)$$

The solution of (5) is:

$$\boxed{\tilde{U}^o(\tau) = 1-e^{-\tau}} \qquad (6)$$

The only dimensionless parameter in the representation of $\tilde{U}^o(\tau)$ is the relative time τ. The absence of ρ from (5) and (6) indicates that the growth of residual radiant energy is basically independent of the medium in which it takes place. At any rate $\tilde{U}^o(\tau)$ will be seen to differ from $\tilde{U}^n(\tau)$, e.g., the growth and decay of which depends critically on the parameter ρ.

Dimensionless Forms for $U^n(t)$

Starting with (24) of Sec. 5.8 under the standard growth condition, we have:

$$\frac{dU^n(t)}{dt} = - \frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s}$$

which we may write as:

$$T_\alpha \frac{[d\ U^n(t)/U^o(\infty)]}{dt} = - U^n(t)/U^o(\infty) + \frac{T_\alpha}{T_s} U^{n-1}(t)/U^o(\infty) \quad ,$$

which by conversion rules 1 and 2 become:

$$\boxed{\frac{d\tilde{U}^n(\tau)}{d\tau} = - \tilde{U}^n(\tau) + \rho\tilde{U}^{n-1}(\tau)} \qquad (7)$$

which has the solution:

$$\boxed{\tilde{U}^n(\tau) = \rho^n \; [1-F_n(\tau)]} \tag{8}$$

where F_n is defined in (13) of Sec. 5.9. From (8) we have immediately that:

$$\tilde{U}^n(\infty) = \rho^n \tag{9}$$

for every $n \geq 1$, and a study of (7) shows that this relation holds also for $n = 0$.

It is interesting to note how (7), even though defined only for $n \geq 1$, actually reduces to the correct relation when $n = 0$. A comparison of (5) and (7), suggests that we can identify the term $\rho\tilde{U}^{n-1}(\tau)$ with 1 when $n = 0$, i.e., we are encouraged to extend the meaning of $\tilde{U}^j(\tau)$ to the case where $j = -1$. Thus let us write:

$$"\tilde{U}^{-1}(\tau)" \quad \text{for} \quad 1/\rho \tag{10}$$

In full dimensional form this means that we have the definitional identity:

$$U^{-1}(t) = P_\eta \; T_s \tag{11}$$

With this extension, we may use (7) as the basic n-ary differential equation which then includes (5) as a special case.

Dimensionless Forms for $U^*(t)$

Applying the conversion rules to (11) of Sec. 5.10, we have, under the standard growth condition:

$$\boxed{\frac{d\tilde{U}^*(\tau)}{d\tau} = -\,(1-\rho)\;\tilde{U}^*(\tau) + \tilde{U}^o(\tau)} \tag{12}$$

with solution:

$$\boxed{U^*(\tau) = \frac{1}{1-\rho}\left[1-e^{-(1-\rho)\tau}\right] - \left[1-e^{-\tau}\right]} \tag{13}$$

It is interesting to see how (13) predicts the growth of scattered radiant energy in extreme media, i.e., media for which $\rho = 0$ and for which $\rho = 1$, e.g., in purely absorbing and scattering media, respectively. To see this, observe that:

$$\lim_{\rho\to 1} \frac{1-e^{-(1-\rho)\tau}}{1-\rho} = \tau$$

Then we have from (13):

$$\tilde{U}^*(\tau) = (\tau-1) + e^{-\tau} \tag{14}$$

Thus in purely scattering media, at $\tau = 0$, $U^*(0) = 0$, and for very small relative times τ:

$$\tilde{U}^*(\tau) \cong \frac{\tau^2}{2} \quad ,$$

so that $\tilde{U}^*(\tau)$ commences growth parabolically from $\tau = 0$. For somewhat larger τ, $\tilde{U}^*(\tau)$ grows essentially linearly with τ, as might be expected. In the case of the other extreme type of space, the purely absorbing space, i.e., one for which $\rho = 0$, equation (13) predicts $\tilde{U}^*(\tau) = 0$ for every τ, as expected. In general for *normal spaces*, i.e., for spaces in which there is present both scattering and absorption, so that $0 < \rho < 1$, (13) predicts the steady state value of U^* to be

$$\tilde{U}^*(\infty) = \frac{\rho}{1-\rho} \tag{15}$$

This agrees with the natural solution computation based on (9):

$$\tilde{U}^*(\infty) = \sum_{n=1}^{\infty} U^n(\infty) = \sum_{n=1}^{\infty} \rho^n = \frac{\rho}{1-\rho} \tag{16}$$

The growth pattern of $\tilde{U}^*(\tau)$ is relatively interesting because the rate of growth of $\tilde{U}^*(\tau)$ exhibits a maximum at a certain finite time which depends on on ρ. Thus, from (13) we have:

$$\frac{d\tilde{U}^*(\tau)}{d\tau} = (e^{\rho\tau}-1)e^{-\tau} \quad . \tag{17}$$

For normal spaces, i.e., when $0 < \rho < 1$, this rate of growth is zero for $\tau = 0$ and $\tau = \infty$ and positive for all intermediate τ. The τ for maximum growth rate is obtained in the usual manner using calculus, and is of the form τ_{max}, where we have written:

$$\text{"}\tau_{max}\text{"} \quad \text{for} \quad \frac{-\ell n(1-\rho)}{\rho} \tag{18}$$

We shall have occasion to return to this relative time in the discussion below on time constants.

Dimensionless Forms for U(t)

Applying the conversion rules to (27) of Sec. 5.8, we have, under the standard growth condition:

$$\boxed{\frac{d\tilde{U}(\tau)}{d\tau} = -(1-\rho)\,\tilde{U}(t) + 1} \tag{19}$$

whose solution is:

$$\boxed{\tilde{U}(\tau) = \frac{1-e^{-(1-\rho)\tau}}{1-\rho}} \tag{20}$$

Note that for purely scattering media ($\rho = 1$):

$$\frac{d\tilde{U}(\tau)}{d\tau} = 1$$

which implies:

$$\tilde{U}(\tau) = \tau$$

for all $\tau \geq 0$. For purely absorbing media, $\tilde{U}(\tau) = \tilde{U}^o(\tau)$. In normal spaces the steady state value of $\tilde{U}(\tau)$ is:

$$\tilde{U}(\infty) = \frac{1}{1-\rho} \tag{21}$$

Dimensionless Forms for $U(t;\alpha)$, $U(t;a)$

From (9) of Sec. 5.10 and the conversion rules we obtain:

$$\boxed{\frac{d\tilde{U}(\tau;\alpha)}{d\tau} = \tilde{U}^o(\tau)} \tag{22}$$

whence, under standard growth conditions:

$$\boxed{\tilde{U}(\tau;\alpha) = (\tau-1) + e^{-\tau}} \tag{23}$$

This agrees with the special case (14) of the representation of $\tilde{U}^*(\tau)$ (alias $\tilde{U}(\tau;s)$), i.e., under the special case where $s = \alpha$. Finally, from (16) of Sec. 5.10:

$$\boxed{\frac{d\tilde{U}(\tau;a)}{d\tau} = (1-\rho)\,\tilde{U}(\tau)} \tag{24}$$

whence, under standard growth conditions:

$$\boxed{\tilde{U}(\tau;a) = \tau - \tilde{U}(\tau)} \quad . \tag{25}$$

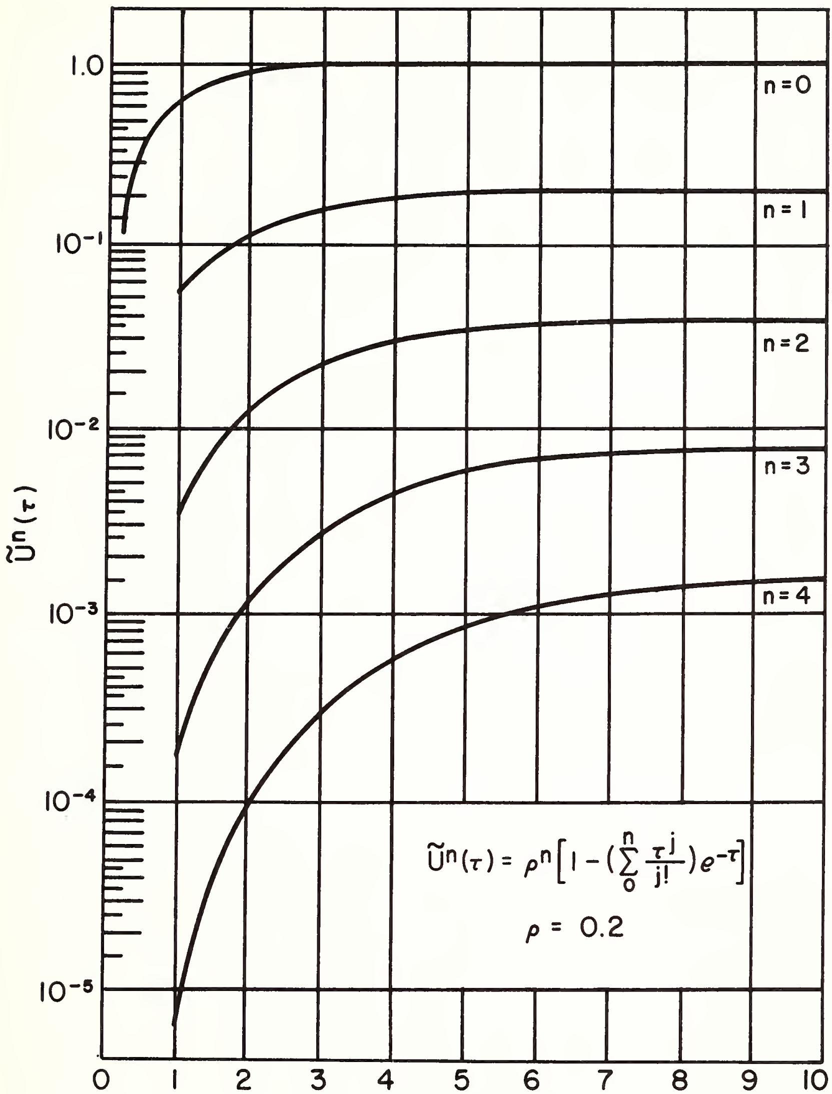

FIG. 5.13 A plot of $\tilde{U}^n(\tau)$ versus τ for n = 0, 1, 2, 3, 4 in an optical medium which has ρ = 0.2 (see (8) of Sec. 5.11).

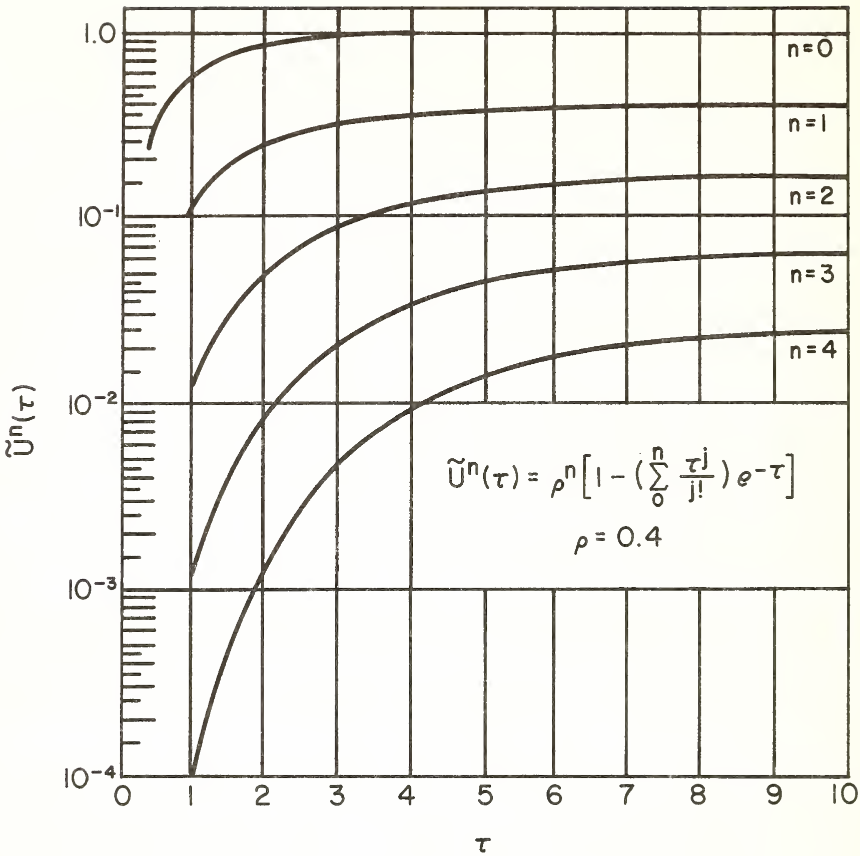

FIG. 5.14 A plot of $\tilde{U}^n(\tau)$ versus τ for n = 0, 1, 2, 3, 4 in an optical medium which has $\rho = 0.4$ (see (8) of Sec. 5.11). Note that the vertical spread of the curves is decreasing, and that the steady state values of $\tilde{U}^n(\tau)$ crowd closer together for higher ρ values.

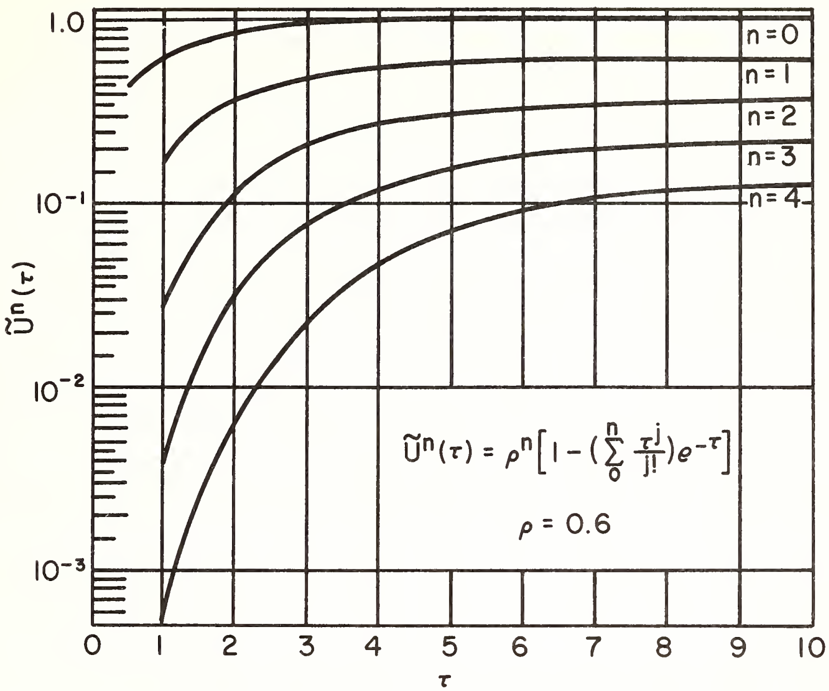

FIG. 5.15 Continuation of Figures 5.13, 5.14.

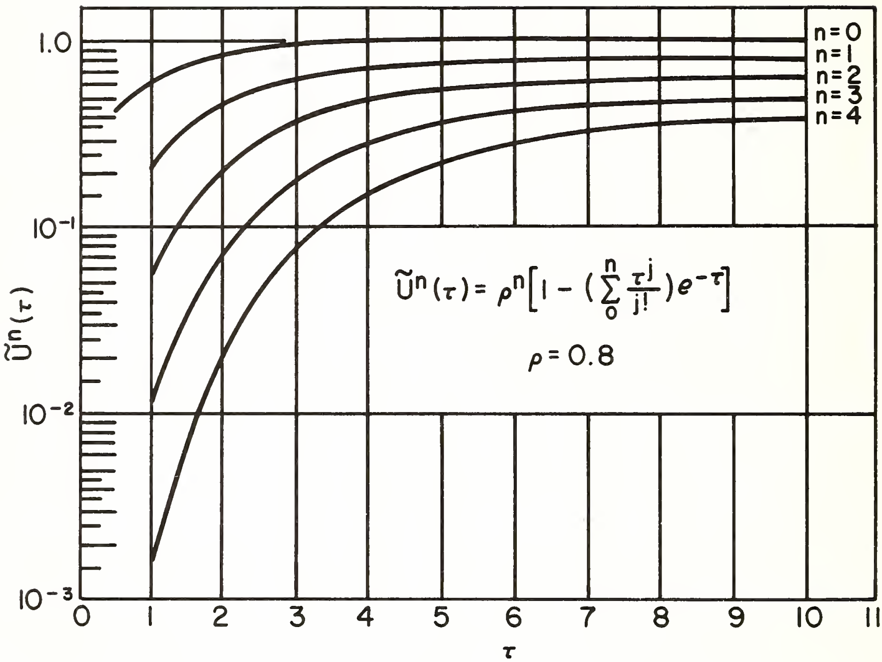

FIG. 5.16 Continuation of Figs. 5.13 through 5.15.

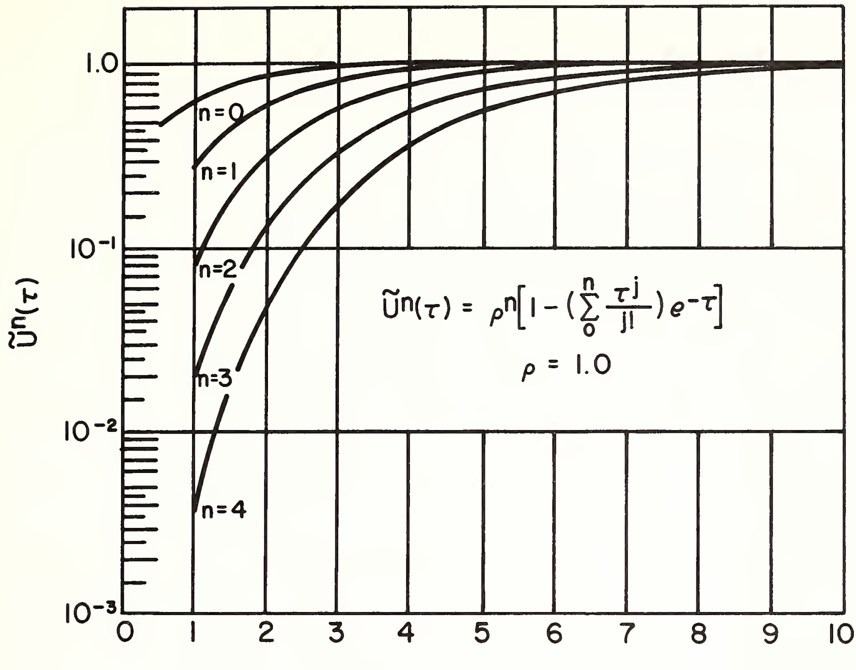

FIG. 5.17 Conclusion of Figs. 5.13 through 5.16.

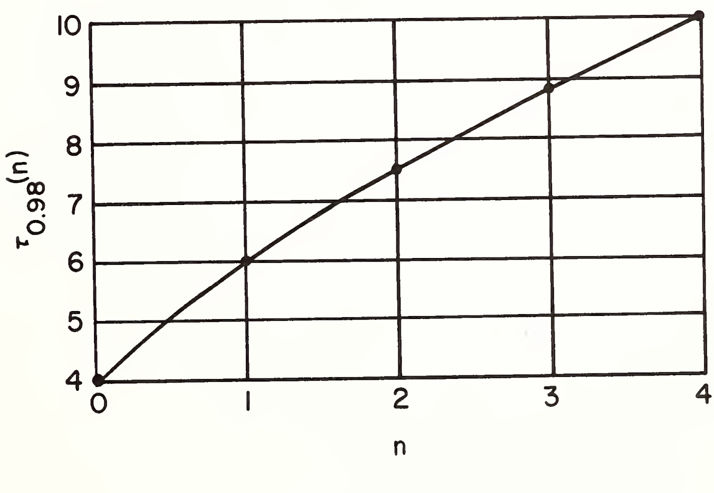

FIG. 5.18 A plot of time constants for $\tilde{U}^n(T)$, n = 0,1,2,3,4 in which c = 0.98. (See (27) of Sec. 5.11.)

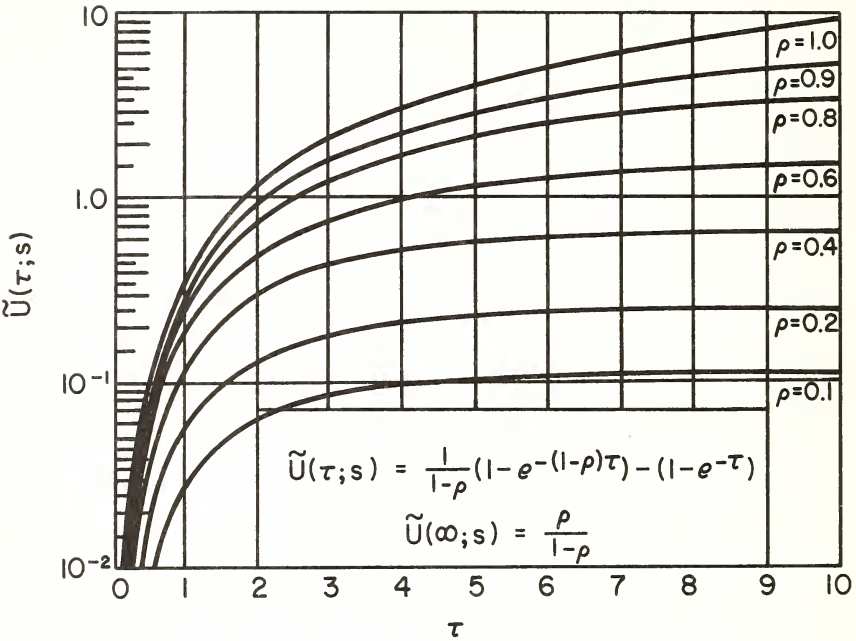

FIG. 5.19 Plots of $\tilde{U}(\tau;s)$ $(=\tilde{U}^*(\tau))$ versus relative time τ. Each curve represents a different scattering attenuation ratio ρ. $\tilde{U}(\tau;s)$ is the dimensionless form of $U(t;s)$, and this latter quantity is the total amount of scattered radiant energy in the optical medium at time t after the steady source has been turned on. $U(t;s)$ is the sum of all n-ary radiant energy components $U^n(t)$, $n = 1,2,3,\ldots$. Some of the latter quantities are plotted in Figs. 5.13 through 5.17, in dimensionless form. Each curve in the present figure, except for $\rho = 1$, levels off to approach the asymptote $\rho/(1-\rho)$. (See (15) of Sec. 5.11.)

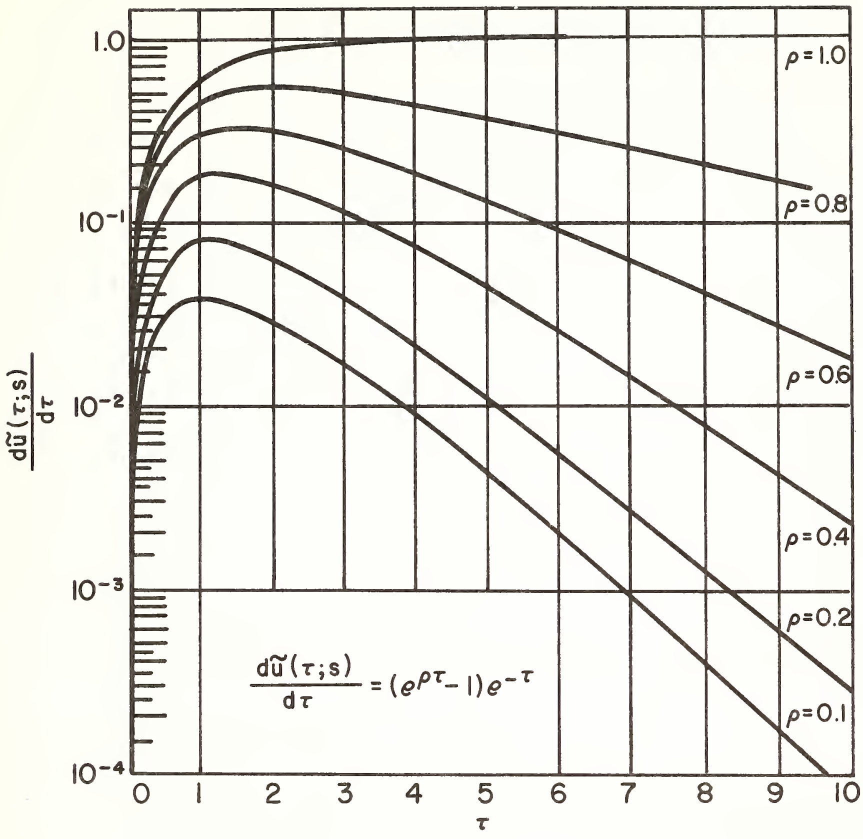

FIG. 5.20 Showing the evolution, in time, of the scattered radiant energy (see (17) of Sec. 5.11).

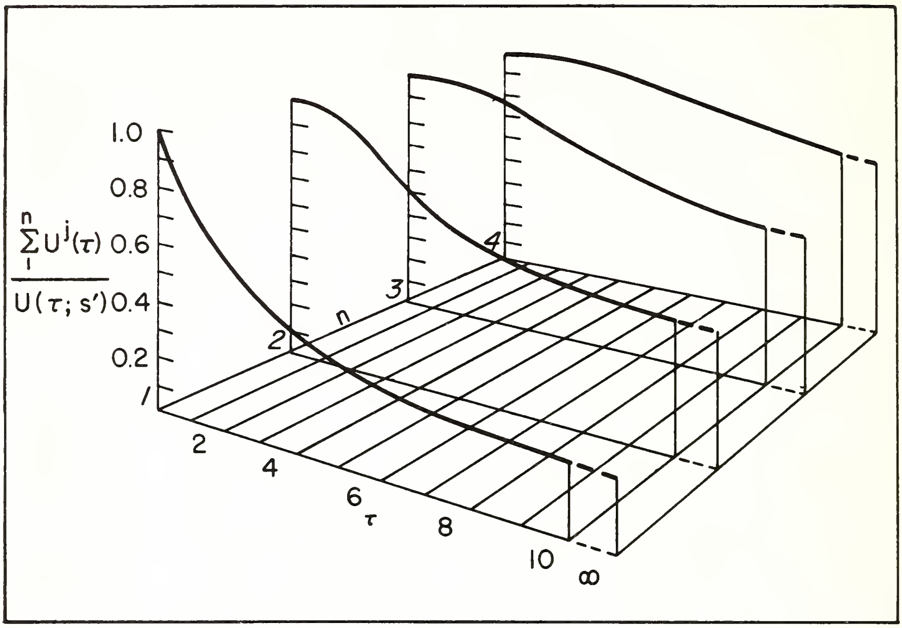

FIG. 5.21 A plot showing the relative magnitude of the sum of the first n scattering orders

$$\sum_{j=1}^{n} U^j(\tau)$$

of radiant energy at time τ as compared to the total amount $U(\tau;s)$ of scattered radiant energy at the same time. The plot is for a space with scattering-attenuation ratio ρ = 0.8. Observe that for fixed n, the ratio is monotonic *decreasing* with time τ. For fixed time τ, the ratio *increases* with increasing scattering order. As an example, let n = 3, and τ = 5. Then the ratio of $U^j(\tau)$ to $U(\tau;s)$ is 0.8; for τ = 10, the ratio is 0.6; and in the limit, as $\tau \to \infty$, the ratio is 0.48. Hence, at steady state the amount of radiant energy having been scattered, once, twice, or three times is 48 percent (= $1 - \rho^n$) of all that has been scattered in general (see Fig. 5.22).

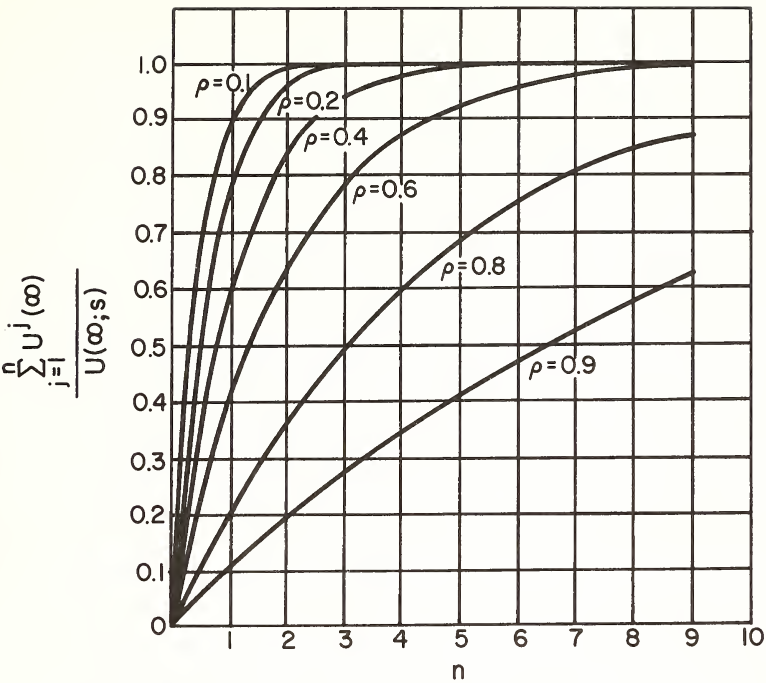

FIG. 5.22 The limiting values, for $\tau = \infty$, of the ratios in Fig. 5.21.

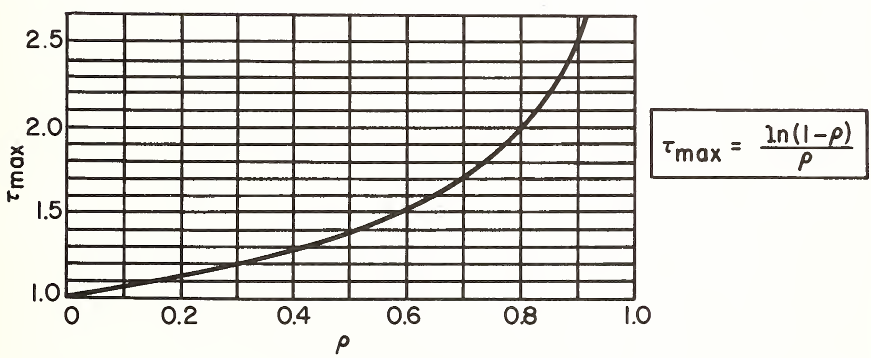

FIG. 5.23 The relative times for the occurrences of the maxima in Fig. 5.20, plotted as a function of ρ. For example, the curve labeled "$\rho = 0.08$" in Fig. 5.20 has its maximum at about $\tau = 2$.

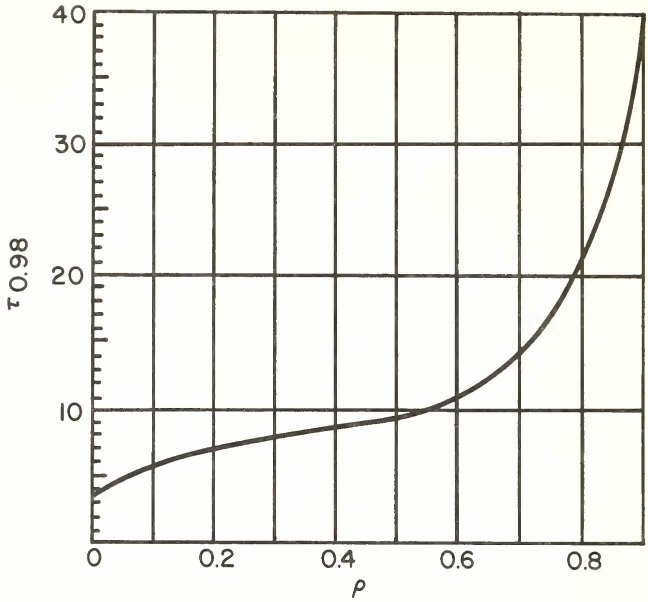

FIG. 5.24 The time constant $\tau_{0.98}$ as a function of scattering-attenuation ratio ρ. (See (26) of Sec. 5.11.)

A Discussion of Time Constants

Time-dependent natural phenomena may be broadly classed into two main groups: those that are periodic and those that are not periodic over a given time interval. Periodic phenomena can in turn be characterized in part by means of their *periods*, i.e., the smallest intervals of time over which they exhibit a basic cycle of behavior. Nonperiodic phenomena on the other hand have very many ways of being nonperiodic, and there is no simple single number which suggests itself as a suitable measure of such general nonperiodicities. Of the great variety of nonperiodic phenomena, however, there are those which appear to eventually tend with increasing time toward a well-defined limit. These *nonperiodic limiting* phenomena can then be characterized in a manner analogous to the periodic phenomena, i.e., by means of single numbers which suitably measure such simple nonperiodicities. One useful means is the concept of the *time constant* of such phenomena. The time constant, broadly speaking, is that interval of time over which the nonperiodic limiting phenomenon evolves from some standard initial state until it arrives just within a prescribed "distance" of its limit state.

Time-dependent light fields in natural optical media are generally phenomena of the *nonperiodic limiting* type discussed above. Therefore the notion of a time constant characterization of such phenomena seems worthwhile exploring. In the discussion that follows we shall examine some possible candidates for time constants of transient light fields in natural optical media. One major fact that will emerge from

the discussion is that there is a large number of possible candidates for time constants, each valuable in the context in which it is found and used. Thus it will turn out that, in the long run, no one single time constant will suffice for the description of every instance in the great variety of time-dependent radiant energy fields encountered in the various natural media (oceans, lakes, atmosphere). *The best choice of time constant that can be made will vary jointly with the type of radiometric concept used (radiance, irradiance, or any of the variety of radiant energies discussed so far) and the space in which the light field is evolving.*

To illustrate the thesis just stated, consider once again the residual radiant energy $U^o(t)$ discussed in Sec. 5.8, now in comparison with the directly observable radiant energy $U(t)$. We saw in Sec. 5.8 the exact analogy that held between a simple resistance-capacitance DC circuit and an infinite homogeneous optical medium in which $U^o(t)$ was evolving. This analogy suggested that the candidate for the time constant associated with U^o in the medium was T_α. Comparing the form of $U^o(t)$ with that of $U(t)$ as given in (14) of Sec. 5.10, we see that in the same medium, but now with reference to $U(t)$, the most obvious candidate for the time constant is T_a. Thus by switching from $U^o(t)$ to $U(t)$ the appropriate choice for time constant correspondingly goes from T_α to T_a.

As another illustration of the thesis of this discussion, consider the scattered radiant energy $U^*(t)$ $(=U(t;s))$ as given in (12) of Sec. 5.10 and its dimensionless graphical representation in Fig. 5.19. The steady state value of $\tilde{U}^*(\tau)$ is $\rho/(1-\rho)$ in normal spaces, i.e., spaces in which $0 < \rho < 1$. Figure 5.19 shows how $\tilde{U}^*(\infty)$ approaches this value asymptotically for selected values of ρ. For example, if $\rho = 0.4$ then $\tilde{U}^*(\infty) = 0.4/(1-0.4) = 0.67$. This value has been attained (at least visually, according to the graph) at about eight relative time units. More generally, in a given space with $0 < \rho < 1$, let c be any number such that $0 < c < 1$. Then we require that value τ_c of τ such that:

$$\frac{c\rho}{1-\rho} = \tilde{U}^*(\tau_c)$$

$$= \frac{1}{1-\rho}\left(1-e^{-(1-\rho)\tau_c}\right) - \left(1-e^{-\tau_c}\right) \tag{26}$$

For every ρ, $0 < \rho < 1$, the number τ_c always exists since $\tilde{U}^*(\tau)$ is continuous and increases monotonically toward its limit, and so eventually takes on the value $c\rho/(1-\rho)$ for $0 < c < 1$. A graph of τ_c for $c = 0.98$ is given in Fig. 5.24 as a function of ρ. For example, for $\rho = 0.4$, $\tau_c = 8$, and so we return to the visual estimate given above. The graph of Fig. 5.24 shows generally that the greater the scattering attenuation ratio, the greater $\tau_{0.98}$--this much could be guessed on intuitive grounds--however, the exact quantitative manner of the increase in $\tau_{0.98}$ is interesting to observe. The numbers τ_c, therefore, can serve as time constants for scattered radiant energy after a choice of c is made.

The time-dependent structure of the scattered radiant energy $\tilde{U}^*(\tau)$ has an additional feature to that of asymptoticity which may serve to be a workable basis for the definition of a time constant. A study of the rate of growth of $\tilde{U}^*(\tau)$ in Sec. 5.11 showed that the derivative of the rate of growth starts out positive, becomes zero at relative time $-\ln(1-\rho)/\rho$, and then remains negative for all subsequent relative times in very given normal medium (cf. (18) of sec. 5.11). This suggests that τ_{max}, the relative time of the maximum rate of growth, is a possible candidate for a time constant for a given medium, for it defines a distinguishable point of inflection on the growth curve of $\tilde{U}^*(\tau)$. Figure 5.23 depicts τ_{max} as a function of ρ for a selected range of normal spaces. The point to observe here is that we need not always base time constant definitions on the feature of asymptoticity of a nonperiodic phenomenon. Well-defined maxima or minima or points of inflection of growth curves may also serve as adequate bases for time constants.

It is interesting to observe how the notion of a time constant can be extended to each of the n-ary radiant energy fields U^n, $n > 0$. The best candidate for the time constant varies with the scattering order n. Thus, suppose c is any number such that $0 < c < 1$. Let $\tau_c(n)$ be that relative time for which:

$$c\tilde{U}^n(\infty) = \tilde{U}^n(\tau_c(n)) = \rho^n \; [1-F_n(\tau_c(n))]$$

holds. That is we require $\tau_c(n)$ such that:

$$1-c = F_n(\tau_c(n)) \quad . \tag{27}$$

As in the case of (26), $\tau_c(n)$ exists for every $n \geq 1$ and c such that $0 < c < 1$. The basis for this conclusion is property 2 of $U^n(t)$, stated in Sec. 5.10, which implies that $\tilde{U}^n(\tau)$ increases monotonically and continuously to its limit. Figure 5.18 depicts a plot of $\tau_c(n)$ for $c = 0.98$ and $n = 0, 1, 2, 3, 4$.

Still one more variation in the concept of time constant follows from the observation that the curves of $\tilde{U}^n(\tau)$ have inflection points at relative times $\tau = n$. Thus setting:

$$\frac{d^2\tilde{U}^n(\tau)}{d\tau^2} = 0 \quad ,$$

implies

$$\tau = n \tag{28}$$

Hence, as in the case of $\tilde{U}^*(\tau)$, we can use the inflection points as identifiable characteristics of the growth curves of U^n. Observe how the time constants suggested by (28) are independent of ρ, and hence the medium, and depend only on n; yet the similar type of time constant for the sum $\tilde{U}^*$ of t the n-ary fields $\tilde{U}^n$ indeed depends on ρ.

With these illustrations we rest our case concerning the nonexistence of a single universally applicable time constant for characterizing transient light fields in extensive optical media. Perhaps, if a single time constant were demanded which could be pressed into use more often than all the other time constants discussed in the present chapter, then we might tentatively suggest T_α for consideration. For T_α appears quite often in the energy context and most critically in the radiance context of (10) of Sec. 5.7. Furthermore, T_α is based on the one inherent optical property (namely α) of optical media which is the most thoroughly documented and which is the most readily measured member of the basic trio α, σ, a.

Finally, we observe that all our preceding deliberations concerned unbounded media--or very extensive media in which their boundaries played a negligible role. For a discussion of the theory of time constants in bounded media in which the sensitivity of radiometer instruments also plays a role the reader may consult the papers in parts IV, V of [236]. These references are part of a set of five reports in which the main discussion centers on the study of the general *metric properties* of time dependent light fields. The theory of the time constant found in [236] is one of the several applications of the general metric theory developed in the series.

5.12 Global Approximations of General Radiance Fields

In this and the following section some of the theory of time-dependent n-ary radiant energy fields will be applied to two general problems of radiative transfer theory. In the present section attention will be directed to the problem of finding relatively simple approximations of time dependent and steady state radiance fields in optical media. In particular it will be shown how the n-ary radiant energy fields may be used to obtain approximations of the observable radiance field such that the approximations are *exact on a global level* over the given medium.

The precise meaning of this phrase will become clear during the course of the constructions of the approximations, to which we now turn. Unless specifically stated otherwise, all constructions will take place on a general optical medium X with arbitrary source conditions.

We begin with the observation that the operator formula

$$N^n = N^1 S^{n-1} \quad ,$$

based on the theory of Sec. 5.1, suggests the following simple approximation, where we write:

$$\text{"}N_g^n\text{"} \quad \text{for} \quad \frac{U^n}{U^1} N^1 \tag{1}$$

Here U^n, $N \geq 1$, is the n-ary radiant energy in X, and N^1 is the primary radiance function in X. N_g^n is called the *global approximation* of N^n for $n \geq 1$.

The reason for such a name and structure of N_g^n lies in the following observations. Note first that N_g^n has scattering order "dimensions" of n-ary radiance. Next, observe that the global approximation for N^n yields the estimate:

$$\frac{1}{v(x)} \frac{U^n(t)}{U^1(t)} \int_{\Xi} N^1(x,\xi,t)\ d\Omega(\xi)$$

for the radiant density function u in X. If we write "u_g^n" for this function, then we see that:

$$u_g^n = \frac{U^n}{U^1} u^1 \qquad (2)$$

for $n \geq 1$. Finally:

$$\int_X^u u_g^n(x,t)\,dV(x) = \frac{U^n(t)}{U^1(t)} \int_X u^1(x,t)\,dV(x)$$

$$= \frac{U^n(t)}{U^1(t)} \cdot U^1(t)$$

$$= U^n(t)$$

This shows that the approximation N_g^n to N^n has the property:

$$U^n(t) = \int_X \frac{1}{v(x)} \left[\int_{\Xi} N_g^n(x,\xi,t)\ d\Omega(\xi) \right] dV(x) \qquad (3)$$

In other words, N_g^n yields the same radiant energy content of X at each time t as does N^n, the actual n-ary radiance function on X. Thus N_g^n yields an exact prediction of approximation of N^n on an overall (or global) basis. The directional or local structure of N^n is approximated by that of N^1, a relatively easily computed function.

The global approximation of N^n may be used to obtain a global approximation of the directly observable radiance N by means of the natural solution representation of N_g^*, where we have written:

$$\text{"}N_g^*\text{"} \quad \text{for} \quad \sum_{j=1}^{\infty} N_g^j \qquad (4)$$

For, by the definition of the N_g^j we have:

$$N_g^* = \sum_{j=1}^{\infty} N_g^j = \sum_{j=1}^{\infty} \frac{U^j}{U^1} N^1 = \frac{U^*}{U^1} N^1 \quad . \tag{5}$$

The requisite global approximation of N is obtained by writing

$$"N_g" \quad \text{for} \quad N^o + N_g^* \tag{6}$$

It follows that:

$$U(t) = \int_X \frac{1}{v(x)} \left[\int_\Xi N_g(x,\xi,t)\, d\Omega(\xi) \right] dV(x) \tag{7}$$

so that N_g indeed endows X with the same radiant energy content as N, the actual observable radiance function on X. The function N_g may then be used to assign to each x in X, and ξ in Ξ at time t the radiance:

$$\boxed{N_g(x,\xi,t) = N^o(x,\xi,t) + \frac{U^*(t)}{U^1(t)} N^1(x,\xi,t)} \tag{8}$$

where, in case standard growth conditions are in force in X, $U^*(t)$ (alias $U(t;s)$) and $U^1(t)$ are given by (14) of Sec. 5.9 and (12) of Sec. 5.10. In the steady state attained under standard growth conditions, (8) yields:

$$\boxed{N_g(x,\xi) = N^o(x,\xi) + \frac{1}{1-\rho} N^1(x,\xi)} \tag{9}$$

which is defined for $0 < \rho < 1$.

Global Approximations of Higher Order

The global approximation N_g^n in (1) above is but the lowest rung on an infinitely high ladder of global approximations of the radiance function in the medium X. We now formulate the global approximation to N of arbitrarily high order. Thus let us for every $n \geq 1$, write:

$$"N_g^n" \quad \text{for} \quad \frac{U^n}{U^{(k)}} N^{(k)}$$

Here we choose to use the same name "N_g^n" for the approximating function, and we have now written, *ad hoc:*

$$"N^{(k)}" \quad \text{for} \quad \sum_{j=1}^{k} N^j$$

and

$$"U^{(k)}" \quad \text{for} \quad \sum_{j=1}^{k} U^j$$

N_g^n is the *global approximation of the kth order of N^n*. It is easy to verify that N_g^n again is globally exact in the general sense of (3). Defining N_g as in (6) and N_g^* as in (4), now for the kth order context, by stopping the sums in (4) and (6) at $j = k$, it follows that:

$$\boxed{N_g^{(k)}(x,\xi,t) = N^o(x,\xi,t) + \frac{U^*(t)}{U^{(k)}(t)} N^{(k)}(x,\xi,t)} \tag{10}$$

we call $N_g^{(k)}$ in (10) the *global approximation of the kth order of N*. $N_g^{(k)}$ is globally exact in the sense of (7), i.e., using $N_g^{(k)}$ in (7) will yield $U^{(k)}(t)$. Observe that this approximation also has the virtue of converging to N as $k \to \infty$. That is:

$$\lim_{k\to\infty} N_g^{(k)} = N \tag{11}$$

This follows from (10) and the facts that:

$$\lim_{k\to\infty} U^{(k)}(t) = U^*(t) \tag{12}$$

and that:

$$\lim_{k\to\infty} N^{(k)} = N^* \quad .$$

In this way we see that the global approximations to N have one additional property over the truncated solutions of Sec. 5.5, namely the global exactness property. The steady state limit version of (10) attained under standard growth conditions is:

$$\boxed{N_g^{(k)}(x,\xi) = N^o(x,\xi) + \frac{1}{1-\rho^k} N^{(k)}(x,\xi)} \tag{13}$$

and which is defined for $k \geq 1$, and $0 < \rho < 1$. Under standard growth or decay conditions, one may use in (10) the

expressions for $U^*(t)$ and $U^n(t)$, developed in Sec. 5.11, to generate useful approximations to time-dependent radiance fields. First or second order global approximations should suffice for many practical settings.

We note in passing that preliminary and informal numerical studies seem to indicate that the shapes (the directional structure) of N^n appear to be spherical (or very nearly so) when n is larger than some integer p which depends on the medium X and ρ. If this conjecture can be proved in general, (probably by means of the set up in 10.5) then an enormous advance in the practical utility of (13) can be made. This conjecture of the limiting shape of N^n as $n \to \infty$, bears a striking analog to the asymptotic radiance theorem studied elsewhere in this work (cf., e.g., Chapter 10). An important application would be to diffusion theory (see (78) of Sec. 6.6).

5.13 Light Storage Phenomena in Natural Optical Media

The applications of the natural mode of solution of radiative transfer problems in optical media discussed in this chapter will now be concluded with a definition and discussion of the light-storage phenomena in such media.

Everyday Examples of Light Storage

Those who have looked out of a window of an airplane as it descended into a sunbathed cloud layer may recall the sudden transition to a brilliant ambient field of light, and how the sensation of brightness in every direction increased to dazzling proportions as the airplane descended further into the upper regions of the cloud. This phenomenon is but one of many common examples of the storage of light by the mechanism of scattering. One can also see evidence of light storage on overcast nights on the outskirts of large cities: the cloud layer hovering low over the city is deeply and extensively illuminated from the street and building lights below. Flashes of lightning in storm clouds can light up an extensive cloud layer from horizon to horizon even though the actual volume taken up by the network of electrical discharges is a minute fraction of the illuminated volume. Lighthouses on densely fogged nights pour a well-defined beam of light into a surrounding fog with the result that the beam and the lighthouse are imbedded in a field of scattered light which, under suitable conditions, may be observed by approaching mariners far sooner than the light of the revolving beam. As one descends into a lake or the ocean on a sunny day, there is a shallow region near the surface in which the radiance measurably increases with increasing depth for various horizontal and upward-looking lines of sight.

These examples illustrate the phenomenon of the storage of light in scattering media. The sense of the work "storage" is used in its everyday sense: the accumulation or building up of radiant energy in the scattering material that surrounds the source of the energy. If one were to quickly extinguish the light source, the stored light would not immediately disappear with the extinction of the source; rather the scattered

light stored in the earth's atmosphere would take on the order of a score of microseconds to be lost into space, or converted into longer wavelengths of radiation and other forms of energy. The decaying atmospheric light field is like the diminishing reverberation of organ notes in a spacious auditorium in which the acoustical energy is momentarily entrapped and redirected by the walls of the auditorium (cf. Sec. 5.6). In the case of light, the walls of the auditorium are replaced by multitudes of tiny scattering centers comprising clouds, fogs, or parts of the entire atmosphere, and the hydrosphere of the earth: the light impinges on the scattering centers and is redirected again and again by scattering.

Thus, the energy of a pencil of photons, which ordinarily traverses a given volume of empty space in one microsecond, could, in principle, be cycled and recycled within the confines of the volume for a period of several dozens of microseconds before it escapes or is transformed. Therefore, if a continuous steady beam of light is poured into such a volume, the steady state density of scattered light stored within the volume could be tens of times greater than the average density of the light ordinarily within the beam.

Do all these phenomena have a common simple description? Is there a small set of properties of the medium and of the source that, when isolated, can serve as the salient parameters in an analytical description of the stored light field? The answer is 'yes'; the natural mode of analysis of light fields plays an essential role in formulating the details of the answer.

In this section we embark on a preliminary attempt to describe the phenomenon of light storage in precisely defined terms. Once we have decided on an exact radiometric definition of the term "stored light energy," we go on to formulate a simple mathematical model of the light field in a scattering-absorbing medium which can describe how the stored light energy depends on the inherent optical properties of the medium, the geometry of the medium, and the properties of the light source.

It turns out that there are several ways in which we may formulate the description of "stored light energy." The form of the description depends on one's choice of the radiometric quantity used in the description. For example, we find that there is a description associated with the radiometric concept of radiance, another description with irradiance, another with radiant density, and still another with radiant energy.

In the present discussion we will limit our attention to the description of stored light energy exclusively by means of the concept of radiant energy. The resulting description is by far the most natural of all the various possibilities; it is, by a happy coincidence, also the most simple to deal with, and the easiest from which to draw examples.

In the event that more detailed descriptions of storage phenomena than those developed in the present study are ever required, such as n-ary radiance N^n or radiance N, recall that

we have formulated the requisite time-dependent transport equations of these radiometric quantities in Sec. 5.2. Therefore, the work of this section should readily be extended to the radiance case by interested researchers. The investigation of the time-dependent radiant flux problem made in the preceding sections also supplements the results of the present study by providing detailed numerical and graphical illustrations (Figs. 5.13-5.24) of the solutions of the n-ary radiant energy equations, and related radiometric concepts, which play an important role in the storage capacity concept.

Storage Capacity

Let "U" represent the directly observable steady state radiant energy attained in an arbitrary medium X under arbitrary growth conditions; let "U^o" represent the amount of U consisting of *residual radiant energy* from the source (associated with photons which have not yet been scattered or absorbed subsequent to entry into X); and finally, let "U*" represent the amount of U consisting of *scattered radiant energy* within the medium (associated with photons which have undergone at least one scattrring operation). The ratio U^*/U is then a measure of the relative amount of scattered radiant energy in the medium X. It is a number which lies between zero and one and will be referred to as the *storage capacity* of the medium X.

In the case of an infinite homogeneous medium whose steady state light field has been attained under standard growth conditions (Sec. 5.11), the storage capacity has a particularly simple representation in terms of the total volume scattering coefficient s, and the volume attenuation coefficient α of the medium:

$$\text{storage capacity} = \frac{U^*}{U} = \frac{s}{\alpha} = \rho \qquad (1)$$

where ρ is the scattering-attenuation ratio. In the case of nonhomogeneous or finite media, the storage capacity is a more complicated function of ρ and the geometry of the medium. (Examples of more general storage capacity formulas will be given below in (5) and (6).) But even in the present simple context, we gain important insight into storage phenomena in general: the storage capacity depends basically on the *relative* magnitudes of s and α. Thus if we consider two media, one in which $s = 0.01/\text{m}$, $\alpha = 0.02/\text{m}$, and another in which $s = 0.10/\text{m}$, $\alpha = 0.20/\text{m}$, we see that the former medium has an attentuation length of $1/\alpha = 50$ m while the latter while the latter medium is an order of magnitude more optically dense with an attenuation length of $1/\alpha = 5$ m. However, the scattering-attenuation ratio for each medium is $\rho = 0.5$. Thus, despite the great disparity in optical density of these media, their storage capacities have a common value, namely $U^*/U = 0.5$, indicating that in the steady state in each medium, the stored radiant energy (in scattered form) is 50% of the total observable energy within each medium.

Methods of Determining Storage Capacity

The problem of determining the storage capacity of an infinite or very extensive optical medium (one in which the boundaries play a negligible role) is readily solved using the results developed in the preceding sections on n-ary radiant energy. In particular, for homogeneous infinite media, the storage capacity reduces to a very simply obtained single number ρ, as shown above. The number ρ is readily determined in practice by a few local measurements. However, the infinite settings are occasionally inadequate models of real situations. In real media in terrestrial settings we usually dispense with computation programs and go directly to the medium (clouds, lakes, oceans) to perform measurements *in situ* over the given region. By following the definition of storage capacity to the letter, we need only try to measure the radiant energy U^* and U by measuring scalar irradiance at each point throughout the medium and find the quotient U^*/U. However, to probe the medium point by point is always laborious and occasionally impossible. A practicable scheme for measuring storage capacity of real media would be one in which all internal probings are obviated. We thus set up the following problem for study: Is there some way of determining $U^*(X)/U(X)$ for a medium X by limiting all radiometric measurements to the boundary of X? The answer is in the affirmative. We now present the details of a possible empirical procedure leading to the storage capacity of a natural optical medium.

The discussion begins with the steady state version of (24) of Sec. 5.8 applied to a homogeneous, bounded region X of some real optical medium. The incident radiant flux on X is arbitrarily disposed over the boundary and X is assumed to have no internal emission sources. Thus we begin with:

$$0 = -\alpha U^n(X) + sU^{n-1}(X) + \frac{1}{v}\bar{P}^n(X) \tag{2}$$

for $n \geq 1$. Here $\bar{P}^n(X)$ is the net inward radiant n-ary flux across the boundary of X. The n-ary radiant flux is indexed relative to the incident radiant flux on the boundary of the optical medium in which X is located. Thus if the optical medium is the ocean and X is a cube 10 m on a side whose center is located 100 m below the surface, then the n-ary radiant flux in the cube is relative to the incident radiant flux on the surface of the ocean. Summing each side of (2) over all $n \geq 1$:

$$0 = -\alpha \sum_{n=1}^{\infty} U^n(X) + s \sum_{n=1}^{\infty} U^{n-1}(X) + \frac{1}{v} \sum_{n=1}^{\infty} \bar{P}^n(X)$$

Using the natural solution properties this becomes:

$$0 = -\alpha U^*(X) + sU(X) + \frac{1}{v}\bar{P}^*(X) \tag{3}$$

where we have written:

$$\text{"}\bar{P}^*(X)\text{"} \quad \text{for} \quad \sum_{n=1}^{\infty} \bar{P}^n(X) \quad . \tag{4}$$

In accordance with our preceding remarks, we are interested in estimating the quantity $U^*(X)$ with the ultimate goal in mind of estimating the ratio $U^*(X)/U(X)$. But any such estimation must be couched in terms of *observable* or *simply calculable* quantities. $U^*(X)$ is not directly observable; and $U(X)$, while observable, is not simply calculable. (It requires a determination of observable radiant density $u(x)$ at each point x of X.) In casting about for easily observable and simply calculable quantities, the observable net flux $\bar{P}(X)$, the residual net flux $\bar{P}^o(X)$ and the residual energy $U^o(X)$ immediately come to mind. If we can obtain an expression for $U^*(X)/U(X)$ in terms of $\bar{P}(X)$, $\bar{P}^o(X)$ and $U^o(X)$, we will have obtained the best solution possible to the problem of empirically determining the storage capacity of a *finite* homogeneous medium.

It turns out that the characterization of $U^*(X)/U(X)$ in terms of $\bar{P}(X)$, $\bar{P}^o(X)$ and $U^o(X)$ is relatively easy to achieve. Starting with (3), and noting by (33) of Sec. 5.8 that we have:

$$\bar{P}(X) = \bar{P}^o(X) + \bar{P}^*(X),$$

we can recast (3) into the form:

$$-\frac{1}{v}\,\bar{P}(X) + \frac{1}{v}\,\bar{P}^o(X) = -\,aU^*(X) + sU^o(X) \quad .$$

We can then represent the nonobservable $U^*(X)$ in terms of observable and calculable quantities:

$$U^*(X) = \frac{s}{a}\,U^o(X) + \frac{1}{av}\,[\bar{P}(X) - \bar{P}^o(X)]$$

Hence

$$\boxed{\frac{U^*(X)}{U(X)} = 1 - \cfrac{a}{\alpha + \left[\cfrac{\bar{P}(X) - \bar{P}^o(X)}{vU^o(X)}\right]}} \tag{5}$$

Equation (5) gives the desired general formulation of the storage capacity of a *finite* homogeneous medium X in terms of the directly observable net inward flux $\bar{P}(X)$ over the boundary of X, the calculable net inward residual flux $\bar{P}^o(X)$ over the boundary of X, and the calculable residual energy content $U^o(X)$ of X. The volume absorption coefficient a and the volume attenuation coefficient α are the inherent optical properties of X which enter into the calculation and which are assumed known.

It should be remarked that equation (5) is an *exact* and computable formula for the storage capacity $U^*(X)/U(X)$ whenever X is any finite homogeneous medium with $a \geq 0$, irradiated by sources in an arbitrary manner and in which the resultant light field is in steady state. If X is infinite in all directions or very extensive, then it may be that $\overline{P}(X) = \overline{P}^o(X)$, and (5) reduces to (1). The condition $\overline{P}^o(X) = \overline{P}(X)$ means that $\overline{P}(X) = 0$, i.e., that there is no net scattered flux across the boundaries of X. This could happen when the boundaries are infinitely far removed, or when a small volume is deep inside an extensive medium.

Example

To illustrate how (5) is used in particular contexts, consider for example a horizontally extensive cloud stratum, or ocean layer with upper boundary on the surface, which is of finite geometric depth under a clear sunlit sky or clear moonlit sky. To fix ideas, consider the ocean layer. We agree that the principal source of flux is to be the sun or moon, as the case may be, with negligible auxiliary sources associated with the sky and ground (or lower layers in the case of the ocean). Suppose the sun cannot be seen through the given layer as one is looking up from below. It may be checked that the difference $\overline{P}(X) - \overline{P}^o(X)$ in (5) then reduces essentially to $-P^*(X,+)$, where $P^*(X,+)$ is the total net outward rate of flow of stored energy across the two boundaries of X. (The inward flow $P^*(X,-)$ is set to zero.) Suppose also that the outward rate of low from X over its lower boundary is small compared to that of its upper boundary (which is compatible with the assumptions above). Then:

$$U^o(X) = \frac{N^o \Omega A}{v\,\alpha \sec\theta} = \frac{P^o(X,-)}{v\alpha}$$

where N^o is the radiance of the sun or moon at the upper boundary of X, θ its angle from the zenith, Ω is its solid angle subtense, and A is the area of the upper boundary of the cloud. The second equality follows from the definition of inward residual flux $P^o(X,-)$ over the upper boundary of X. Hence (5) becomes

$$\boxed{\frac{U^*(X)}{U(X)} = \frac{\rho - R(X)}{1 - R(X)}} \quad , \qquad (6)$$

where "R(X)" stands for $P^*(X,+)/P^o(X,-)$, the reflectance of X at its upper boundary, a directly measurable quantity.

As a simple numerical illustration of (6), suppose that we take the case of a part X of the ocean for which (6) holds and for which it is found that $\rho = 0.4$ and that $R(X) = 0.02$ for a given wavelength of light around the middle of the visible spectrum. Then the storage capacity U^*/U is:

$$\frac{0.4 - 0.02}{1 - 0.02} = \frac{0.38}{0.98} = 0.39$$

If some time later U^o is known to be a certain amount over the same layer, then, if "C" denotes the storage capcity, clearly:

$$\boxed{U = \frac{U^o}{1-C}} \tag{7}$$

and hence the directly observable radiant energy in the layer is estimable from U^o and knowledge of C.

Equations (6) and (7) illustrate but two of the many practical formulas which may be deduced--under various hypotheses--from the exact formula (5). The preceding derivation will suffice to indicate the general outline of such procedures, and we leave the exploration of other possibilities to the interested reader.

5.14 Operator-Theoretic Basis for the Natural Solution Procedure

We close the present chapter with an overview of the theoretical activities of the chapter. As in the earlier general discussions of the canonical equations (Sec. 4.7) the present discussion will perhaps not so much increase our ability to solve specific problems of applied radiative transfer as it will deepen insight into the essential structure of the natural solution procedure, and therefore radiative transfer theory. In particular the general results below will show how radiative transfer theory, via the integral form of the equation of transfer, is connected to those parts of the main stream of mathematical physics which share with the present field certain operator equations whose mode of solution coincides, on the abstract level, with the natural mode of solution studied in this chapter. The discussion is intended to be intuitive, as far as the material will allow.

Let L be a general (not necessarily linear) operator defined on a domain $\mathfrak{D}$ of functions such that Lf is in $\mathfrak{D}$ whenever f is in $\mathfrak{D}$. Thus L maps elements of $\mathfrak{D}$ into $\mathfrak{D}$. Next suppose $\mathfrak{D}$ has a "distance function" d defined on it such that if f and g are in $\mathfrak{D}$, then d(f,g) is a nonnegative real number with the properties:

(i) $d(f,g) = 0$ if and only if $f = g$

(ii) $d(f,g) = d(g,f)$

(iii) $d(f,h) \leq d(f,g) + d(g,h)$

The function d is called a *metric* for $\mathfrak{D}$, and as can be seen, it has the three main properties of ordinary distance relation of everyday life. We summarize all this by saying that the pair $(\mathfrak{D},d)$ is a *metric space*.

Now the connection between $(\mathfrak{D},d)$ and the radiative transfer setting of this chapter is quite easily made. Let X be an optical medium with initial radiance N^o and let S^1 be the operator in (5) of Sec. 5.7. Then write:

$$\text{"L" for } N^{o} + (\cdot)\, s^{1} \tag{1}$$

and we have an example of the operator L above, where $\mathfrak{D}$ is now the set of all radiance functions on X. Thus if N is a radiance function on X (i.e., N has the dimensions of radiance) then certainly

$$N^{o} + Ns^{1}$$

is again radiance function on X. We are not asserting at the moment that N is a solution of the equation of transfer, but merely making an observation that the function displayed above has the dimensions of radiance, and that is all at the moment that is required for admission into $\mathfrak{D}$. Hence L as defined in (1) maps elements of $\mathfrak{D}$ back into $\mathfrak{D}$.

Next we show that there is a very natural counterpart in radiative transfer theory to the abstract metric d for each fixed time t and bounded optical medium X. Let us write

$$\text{"d(f,g)" for } \int_X \frac{1}{v(x)} \left[\int_\Xi |f(x,\xi,t) - g(x,\xi,t)|\, d\Omega(\xi) \right] dV(x) \tag{2}$$

It is easy to verify that if f = g, then d(f,g) = 0, and that if d(f,g) = 0, then f = g except on sets of directions ξ and points x of zero measure. This exception can be smoothed over by advanced technical devices,* and we henceforth can assume condition (i) for a metric to be satisfied. Next one can verify conditions (ii) and (iii) with ease and the verification is left to interested readers. We call the metric function d as defined in (2), the *radiometric*. By various standard techniques (e.g., averaging) (2) can readily be extended to unbounded media. An alternate choice of metric can also be made by writing

$$\text{"d(f,g)" for } \sup_{x,\xi} |f(x,\xi,t) - g(x,\xi,t)| \tag{2a}$$

where

$$\text{"}\sup_{x,\xi} h(x,\xi)\text{"}$$

---

*In particular, this can be done by means of equivalence classes of functions, an equivalence class being the set of all radiance functions on a domain Y which differ from one another at most on subsets of Y of zero measure. Then we go on to work with equivalence sets of functions rather than individual functions. However, for the present we work directly with the radiance functions, with no essential loss of rigor.

means the supremum (the maximum) of the values of $h(x,\xi)$ as x,ξ vary over all permissible values in the domain of h. The function d in (2a) also satisfies all the properties (i) to (iii) of a metric. We shall call d in (2a) the *supremum metric.*

We summarize what has been done so far: The operator (1) associated with the integral equation of transfer of classical radiative transfer theory may be viewed as a special case of an abstract operator L on a metric space $(\mathcal{D},d)$, the particular classical form of the operator being given in (1), with $\mathcal{D}$ being the class of all radiance functions on X, and with d the radiometric as defined in (2) or the supremum metric as given in (2a). In what follows we allow $\mathcal{D}$ to contain negative valued radiance functions as well as nonnegative valued radiance functions. Of course in physically meaningful applications we shall always work with the latter; however, for mathematical purposes it is convenient also to have the former.

We now come to a key property of the radiative transfer operator $\mathbf{S}^1$ which can be abstracted from the setting of the present chapter and carried out far into the reaches of abstract operator theory, where its general utility can be more easily discerned. In Sec. 5.7 we showed that if $\overline{N}$ is an upper bound (or supremum) of a radiance function, then (cf. (7) of Sec. 5.7):

$$(N\mathbf{S}^1)(x,\xi,t) \leq \overline{N}\rho(1-e^{-t/T_\alpha})$$

for every x in X, ξ in Ξ and t in $(0,t)$, where "T_α" stands for $1/v\alpha$. From this we are led to deduce that for every pair f,g of radiance functions, and with the supremum metric (2a).

$$d(Lf,Lg) \leq c\ d(f,g) \tag{3}$$

where c is a number which depends only on t, ρ and T_α, i.e., where we have written:

$$\text{"c" for } \rho(1-e^{-t/T_\alpha}) \quad .$$

In all normal optical media (i.e., for which $0 < \rho < 1$), we have $0 < c < 1$ whenever $t > 0$. The proof of (3) is immediate, using the definitions (1) and (2a). Whenever an operator L on a general metric space $(\mathcal{D},d)$ has property (3), we say that L is a *contraction mapping* or that it has the *contraction property*. Hence our particular classical radiative transfer operator L given in (1) is a contraction mapping, relative to (2a). The reader may show that (3) also holds under suitable conditions, relative to (2).

To summarize our findings so far: The operator L associated with the time-dependent integral equation of transfer may be viewed as a special case of a contraction mapping L on a metric space $(\mathcal{D},d)$.

We now have developed enough abstract machinery to illustrate the essential activity of the natural solution procedure, on a very general level--a level which is in contact

with the general representations of widely different natural phenomena in modern physics. Let us choose any function $f^{(o)}$ in $\mathfrak{D}$ and write:

$$\text{"}f^{(1)}\text{"} \quad \text{for} \quad Lf^{(o)}$$

Thus we operate on $f^{(o)}$ in $\mathfrak{D}$ with L to obtain $f^{(1)}$ in $\mathfrak{D}$. We repeat this operation a finite number of times to obtain $f^{(n)}$ where we have written:

$$\text{"}f^{(n)}\text{"} \quad \text{for} \quad Lf^{(n-1)}$$

In this way we obtain a sequence

$$\left\{f^{(o)},\ f^{(1)},\ \ldots,\ f^{(n)},\ \ldots\right\}$$

of functions in $\mathfrak{D}$. As in the case of Sec. 5.1, we can define iterates L^n of L so that (cf., e.g., (11) of Sec. 5.1):

$$f^{(n)} = L^n f^{(o)}$$

Before going on, the reader should verify that if we use L in (1), and N^o for $f^{(1)}$, then $f^{(n)}$ is simply

$$\sum_{j=0}^{n} N^j \quad ,$$

i.e., the sum of the n-ary radiances up to order n.

Since L is a contraction mapping, we have, for $m \geq n$:

$$\begin{aligned}
d(f^{(n)}, f^{(m)}) &= d(L^n f^{(o)},\ L^m f^{(o)}) \\
&\leq c^n\ d(f,^{(o)},\ L^{m-n}\ f^{(o)}) \\
&\leq c^n \left\{d(f^{(o)}, f^{(1)}) + d(f^{(1)}, f^{(2)}) + \ldots + d(f^{(m-n-1)}, f^{(m-n)})\right\} \\
&\leq c^n\ d(f^{(o)}, f^{(1)}) \left\{1 + c + c^2 + \ldots + c^{m-n-1}\right\} \\
&\leq \frac{c^n\ d(f^{(o)}, f^{(1)})}{(1 - c)}
\end{aligned} \tag{4}$$

Since c is less than 1, c^n is arbitrarily small for sufficiently large n. Thus the sequence

$$\left\{ f^{(o)},\ f^{(1)},\ \ldots,\ f^{(n)},\ \ldots\right\}$$

constructed above is a Cauchy sequence (in the sense of modern calculus). By establishing this feature of the sequence we have reached the penultimate step in our general discussion of the natural solution procedure.

The significance of the Cauchy sequence feature of

$$\{ f^{(o)}, f^{(1)}, \ldots, f^{(n)}, \ldots \}$$

is this: In all physically meaningful settings for the metric space $(\mathfrak{D},d)$, it is possible to arrange matters so that, whenever a sequence

$$\left\{ f^{(n)} \right\}$$

of elements in $\mathfrak{D}$ is a Cauchy sequence in the sense of (4), *then that sequence has a limit in* $\mathfrak{D}$. In general, whenever a metric space $(\mathfrak{D},d)$ has this property, we say that $(\mathfrak{D},d)$ is *complete*. It is easy to show that all physically meaningful radiative transfer settings always can be represented by complete metric spaces $(\mathfrak{D},d)$. Let us assume therefore for the remainder of the discussion that $(\mathfrak{D},d)$ is complete.

Taking up the thread of the argument at (4) we now can assert the existence of a limit function f to the sequence constructed above. Thus let us write:

$$\text{"}f\text{" for } \lim_n f^{(n)} \tag{5}$$

We now show that f has two very important properties:

(i) f satisfies the operator equation $f = Lf$

(ii) f is the only function in $\mathfrak{D}$ for which (i) holds, i.e., if $g = Lg$ and $f = Lf$, then $f = g$.

Property (i) follows readily by noting that, by definition, $f^{(n)} = Lf^{(n-1)}$. Hence applying the limit operation to each side of this identity, the result follows by observing that L is a continous mapping* (so that the limit operation can be pushed past L and made to act directly on $f^{(n-1)}$). Property (ii) follows from (i) and the contraction property of L:

$$d(f,g) = d(Lf,Lg) \leq c\, d(f,g)$$

From this (since $c < 1$) we must have $d(f,g) = 0$, so that $f = g$.

Let us now make the final summary of what has been done so far in this section: The natural mode of solution in radiative transfer theory has been found to take its place as a special case of a very general operator technique in modern functional analysis. This technique is based on the following theorem (cf., e.g., [140]):

Theorem (Principle of Contraction Mappings). Every contraction mapping L on a complete metric space $(\mathfrak{D},d)$ generates one and only one solution of the equation $f = Lf$.

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*A point which is readily established in functional analysis texts (cf., e.g., [140]).

The classical radiative transfer setting entities are paired off with the abstract setting entities of the preceding theorem as follows:

	In Radiative Transfer Theory	In the Theorem
a)	Set $\mathscr{D}$ of all radiance functions on an optical medium X	$\mathscr{D}$
b)	The radiometric d, as in (2) or (2a)	d
c)	The operator L, as in (1)	L

We will make one final remark on the existence of the solution f of the general operator equation f = Lf. This is the observation that the solution f defined in (5) is independent of the initial function $f^{(o)}$ starting the chain of iterations $L^n f^{(o)}$. This fact becomes clear, at least logically, by noting the uniqueness property (ii) above. For if $f^{(o)}$ and $g^{(o)}$ are two distinct initial functions, then construction of their iteration sequences yields f and g such that property (i) holds for each.

5.15 Bibliographic Notes for Chapter 5

The natural mode of solution of the equation of transfer studied in this chapter, as noted in the introduction, plays a unique, fundamental role in radiative transfer theory. The formal power of the method and its intuitive simplicity cannot be overemphasized. For some historical notes on the natural mode of solution, see Secs. 26 and 42 of Ref. [251]. For recent modifications of the iterative concept of solutions of functional equations, especially for numerical purposes, see [171].

The development of the natural solution, as presented in Secs. 5.1 and 5.4, follows in the main that given in Ref. [251]. The canonical representation of primary radiance in (8) or (9) of Sec. 5.3 is occasionally referred to as "Seeliger's formula," and is to be conceptually distinguished from the more useful and accurate representation of N_r^* given in (5) of Sec. 4.4. The only common feature of the two radiance representations is that they both fall within the purview of the basic canonical formula (4) of Sec. 4.7.

The discussion of the "optical ringing problem" in Secs. 5.7 and 5.8 is based on the natural-solution approach to the time-dependent radiative transfer problem, and is designed to be more precise than simple time-dependent *classical* diffusion theory (Sec. 6.6). The approach outlined in these sections is drawn from the results in Ref. [211]. A related approach to the optical ringing problem from the point of view of temporal metric spaces was tentatively explored in the series of reports [236]. Further approaches to time-dependent radiative transfer problem are possible via the higher-order diffusion equations. See Table 1 of Sec. 6.5. The truncated natural-solution inequalities in Sec. 5.7 are based on [239]. Further inequalities in this circle of ideas may be found in Ref. [67].

The material of Secs. 5.8 to 5.12 is drawn, with minor revisions, from Ref. [211]. The light storage discussions in Sec. 5.13 are based on Ref. [237]. The abstract overview of the natural mode of solution in Sec. 5.14 uses advanced concepts of functional analysis (in particular, the principle of contraction mappings) which may, e.g., be studied in Ref. [140].

In the opening remarks of Sec. 5.11, it was emphasized that the dimensionless forms of the equations describing n-ary radiant energy fields are shared by many natural processes, some quite distinct conceptually from the time-dependent evolution of radiant energy in optical media. For a brief exploration of such alternate processes governed by the same equations, see Chapter 14 of Ref. [39] and the footnotes in that chapter.

The analogies between radiative transfer phenomena and other transport phenomena discussed in Sec. 5.11 also can be pursued further, e.g., in [259] and [312].

CHAPTER 6

CLASSICAL SOLUTIONS OF THE EQUATION OF TRANSFER

6.0 Introduction

In this chapter we shall conduct an exposition of the two most important classical modes of solution of the equation of transfer used in practice besides the canonical and the natural modes discussed in the preceding two chapters. These classical modes are the powerful spherical harmonic method, and the mathematically interesting diffusion method. The spherical harmonic method is classical in the sense that it dates back to Eddington and Jeans [120], two of the pioneers of radiative transfer theory. The spherical harmonic method represents radiance functions in terms of sums of products of two factors: one factor being purely spatial, the other purely directional, an intuitively natural representation for functions defined on the phase space $X \times \Xi$. On the other hand, there are two main theories of diffusion: the classical and the exact theories. The classical diffusion method is based on Fick's law and views photons in optical media as swarms of particles diffusing with great speed, but generally in the manner of classical diffusion processes, such as heat conduction and Brownian motion. The exact diffusion method, which in its essential modern form dates back to the work of Hopf [111], transcends in accuracy the classical diffusion method but is less general in applicability than the spherical harmonic method, in that it applies strictly only to general transport media whose volume scattering function values $\sigma(x;\xi';\xi)$ are independent of the directions ξ' and ξ. However, the relatively great tractability of the equation of transfer resulting from the introduction of this simplification has led to many interesting and fairly detailed exact solutions of the transfer equation, some of which are quite valuable in practice. For this reason we include in our present discussions a brief exposition of the two main diffusion methods. Together, the spherical harmonic method and the diffusion methods form useful adjuncts to the basic natural mode of solution and the canonical mode of solution discussed earlier in this work.

The plan of the chapter is as follows: We begin with the spherical harmonic method. To show the extraordinarily wide scope and power of the method and also its inherent simplicity we derive it in much more general settings than is customary, and from an abstract algebraic point of view. This will be done in Sec. 6.2, after a preliminary section devoted

to motivating the method. Then follows a specialized development of the method using the functions which have given the method its name (Sec. 6.3) but which, in view of the exposition of Sec. 6.2, need no longer exclusively be used. An illustrative example of the spherical harmonic method is given in Sec. 6.4 for plane-parallel media. The discussion of the algebraic idea underlying the spherical harmonic method will be taken up again as a matter of course in Chapter 7 wherein we shall view the method from a more fundamental point of view, namely from the viewpoint of the generalized invariant imbedding relation (Sec. 7.10). In Sec. 6.5, we turn to the diffusion methods, developing them directly from the equation of transfer by imposing the characteristic assumptions of each theory into the equation. The solutions of some of the more famous models in the classical diffusion method are discussed in Sec. 6.6. In Sec. 6.7 the Milne model for infinite media with point sources is discussed, followed by some relatively recent results on a related problem on point source problems in semi-infinite media. The chapter is concluded in Sec. 6.8 by a brief bibliographic survey of other classical methods of solution comprising some of the stock in trade of current radiative transfer theory.

6.1 The Bases of the Spherical Harmonic Method

In this section we shall describe the physical and mathematical bases of the spherical harmonic method. We begin with a brief discussion of the motivation for factoring the radiance function values $N(x,\xi)$ into a sum of products of the form: $f(x)g(\xi)$. We then go on to show how this intuitively and physically natural decomposition is sanctioned and given a direct representation in terms of vector space theory. To accomplish this program, the mathematical prerequisites will entail no more than standard advanced calculus techniques.

Physical Motivations

The steady state radiance function is essentially a function of two variables: the spatial variable x and the directional variable ξ. When one examines the equation of transfer, in either its integrodifferential or integral forms, one is confronted with the complicating presence of the integral term--which represents an integration over the directional variable. If it weren't for that integral term, the equation of transfer would be a simple differential equation and the theory would long ago have been worked out and forgotten by mathematicians! When an investigator, new to the field of radiative transfer theory, encounters the equation of transfer, one of his more probable actions would be to see what would happen if the radiance function N is assumed to be the product of two functions f and g, such that:

$$N(x,\xi) = f(x)g(\xi) \quad . \tag{1}$$

Could the radiance function in some optical media be represented simply as such a product? It would be instructive to follow the consequences of this query, as it is at once one

of the most natural and fruitful of questions to investigate in the task of solving transfer problems.

The immediate effect of such an assumption as (1) would be the reduction of the path function N_* to the form:

$$N_*(x,\xi) = \int_{\Xi} N(x,\xi') \, \sigma(x;\xi';\xi) \, d\Omega(\xi)$$

$$= f(x) \int_{\Xi} g(\xi') \, \sigma(x;\xi';\xi) \, d\Omega(\xi') \quad . \qquad (2)$$

It looks as if the assumption (1) is ineffective unless a similar assumption is made about the volume scattering function. Thus, in the spirit of (1), another assumption is made, now about σ: We assume that two functions c and p exist and are such that:

$$\sigma(x;\xi';\xi) = c(x) p(\xi';\xi) \qquad (3)$$

Using (3) in (2), the representation of $N_*(x,\xi)$ becomes:

$$N_*(x,\xi) = f(x) \; c(x) \int_{\Xi} g(\xi') \, p(\xi';\xi) \, d\Omega(\xi')$$

$$= f_*(x) \; g_*(\xi) \qquad (4)$$

where f_* and g_* are defined in the obvious way. Therefore, under the additional assumption (3), the path function N_* may, like N itself be represented as a product of two functions: one of x alone, the other of ξ alone.

The next step in the explorations would be to see if the equation of transfer becomes more tractable with (1) and (3) as starting points. Thus, starting with the equation of transfer:

$$\xi \cdot \nabla N(x,\xi) = \frac{dN(x,\xi)}{dr} = - \alpha(x,\xi) \, N(x,\xi) + N_*(x,\xi) \quad , \qquad (5)$$

and using (1) and (3), the equation becomes:

$$g(\xi) \frac{df(x)}{dr} = - \alpha(x,\xi) \, f(x) \, g(\xi) + f_*(x) \, g_*(\xi) \qquad (6)$$

Having split apart the spatial and directional components of σ, as shown in (3), it is physically reasonable (but not

logically necessary) to do likewise with α. Succumbing for the moment to physical reasonability, so that the discussion can proceed, we assume $\alpha(x,\cdot)$ to be constant valued on Ξ for every x in X, and write simply "$\alpha(x)$" for this common value at x. Then (6) can be rearranged into the form:

$$\frac{dr(x)}{dr} = f(x) \left[c(x)\ g_*(\xi)/g(\xi) - \alpha(x) \right] \tag{7}$$

Two observations may now be made. First, the results of the accumulated assumptions, succinctly residing in (7), show that f(x) is in principle determinable by a simple integration of the differential equation (7) along a path of sight provided the values of the parenthesized terms in (7) are known. The second observation is that the values of the parenthesized terms in (7) are known once the quotient $g_*(\xi)/g(\xi)$ is known. By an inspection of (7), it is clear that this quotient must be some number independent of ξ. Hence we write:

$$g_*(\xi)/g(\xi) = \lambda \quad , \tag{8}$$

which then in turn requires the function g to satisfy the integral equation of the form:

$$\lambda g(\xi) = \int_{\Xi} g(\xi')\ p(\xi';\xi)\ d\Omega(\xi') \tag{9}$$

The net result of the assumptions (1) and (3) are to reduce the problem of the solution of (5) into subproblems: the solution of an integral equation (9) for g, with an appropriate λ; and a solution of the simple ordinary differential equation (7) for f, using the λ obtained in process of finding g.

It appears therefore that up to this point a definite step has been made in the solution of (5) by adopting the assumptions (1) and (3). It seems worthwhile to follow this promising start and to attempt to carry the solution of (9) to completion. If this can be done for all physically reasonable assumptions on $p(\xi';\xi)$ in (3), then a general solution of the equation of transfer will have been found. Toward this end we will adopt for $p(\xi';\xi)$ the property of (*weak*) *isotropy*, i.e., the property that for every ξ' and ξ:

$$\int_{\Xi} p(\xi';\xi)\ d\Omega(\xi) = \int_{\Xi} p(\xi';\xi)\ d\Omega(\xi') \quad .$$

Since either integral will be independent of ξ, or ξ', we shall set its fixed value equal to 1. This puts the burden of the correct magnitude of σ on c(x) in (3). In fact we now see that c(x) is none other than the volume total scattering

value at x in X because, by (3) of Sec. 4.2:

$$s(x;\xi') = \int_{\Xi} \sigma(x;\xi';\xi)\ d\Omega(\xi) = c(x) \int_{\Xi} p(\xi';\xi)\ d\Omega(\xi)$$

$$= c(x) \quad .$$

Hence $s(x;\xi')$ is independent of ξ', and we write "s(x)" for this common value at x. In this way we simultaneously normalize p and give c a physical interpretation.

A similar normalization can be made of g in (1) with the corresponding effect of giving f a convenient physical interpretation. Thus, requiring g to have the property:

$$\int_{\Xi} g(\xi)\ d\Omega(\xi) = 1$$

it follows from (1) that:

$$h(x) = \int_{\Xi} N(x,\xi)\ d\Omega(\xi) = f(x) \int_{\Xi} g(\xi)\ d\Omega(\xi) = f(x) \quad .$$

Hence f is in this case simply the scalar irradiance function h.

Returning now to the two reduced equations (7) and (9) we have from (7) and (8) that:

$$\frac{dh(x)}{dr} = h(x) \left(\lambda s(x) - \alpha(x) \right) \tag{10}$$

Furthermore, from (9), by integrating each side over Ξ, we find:

$$\lambda \int g(\xi)\ d\Omega(\xi) = \int_{\Xi} g(\xi') \left[\int_{\Xi} p(\xi';\xi)\ d\Omega(\xi) \right] d\Omega(\xi')$$

whence

$$\lambda = 1 \tag{11}$$

so that (10) reduces to:

$$\frac{dh(x)}{dr} = - a(x)h(x) \tag{12}$$

and (9) becomes:

$$g(\xi) = \int_{\Xi} g(\xi') \; p(\xi';\xi) \; d\Omega(\xi') \tag{13}$$

We now have reduced the problem of determining the radiance function N, under the assumptions (1) and (3), to the problem of a simple integration of (12) along a path with respect to path length r, and the solution of (13). The solution of equation (12) presents no difficulty, the general solution being:

$$h(x) = h(x_0) \exp\left\{ - \int_0^r a(x') \, dr' \right\} \tag{14}$$

when the integration is taken along a straight path $\mathcal{P}_r(x_0,\xi)$ of length r from point x_0 to x. The intermediate point x' is the form $x_0 + r'\xi$, $0 \leq r' \leq r$.

Finally, we turn to (13) and immediately observe that any constant function on Ξ, whose value for every ξ in Ξ is some arbitrary fixed value g_0, is a solution. It follows that, if g is any nonconstant solution of (13) then so will $g + g_0$ be a solution of (13). This nonuniqueness of solutions of (13) is a most undesirable state of affairs for a physical model of the light field. This means that, on physical grounds, we must generally reject the model constituted by equations (12) and (13). It follows further that we must reject either or both assumptions (1), (3) which gave rise to (12) and (13). Since (3) is quite tenable on physical grounds, it follows that we must generally* reject (1). In this way we have shown that the initial attempt to factor N into a product of a scalar irradiance function h and a directional function g is untenable on physical grounds. By repeating the essential steps of the arguments between (1) and (13) the same negative conclusion may be deduced for the case where N is represented as a finite sum of terms of the form $h_i g_i$.

The intuitive concept of factoring N into spatial and directional components in general media has thus been shown to be unsupportable on practical physical grounds. However, the factoring may be possible in certain geometrically and physically ideal media. Indeed, as we saw in Sec. 4.4, plane-parallel media with uniform volume scattering functions permit such a factoring of N. According to (9) of Sec. 4.4, we

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*In particular, if a g can be found which satisfies (13), then some approximate models may be found by adjusting g_0 empirically in $N(x,\xi) = h(x) \, (g(\xi) + g_0(\xi))$.

have

$$g(\xi) = \frac{\rho}{4\pi} \frac{1}{1 + \left(\frac{K}{\alpha}\right)\cos\theta} \tag{15}$$

where K is now determined by the requirement that the normalization property of g holds. Thus by adding two more assumptions to (1) and (3), namely that h(x) varied exponentially with a certain fixed exponential decay rate K, and that $\sigma(x;\xi';\xi)$ is independent of ξ' and ξ, a very special factorable radiance function is forthcoming.

The additional physical conditions of the required special exponential character of h and the uniform directional structure of σ are quite severe restrictions to impose on general media in order to obtain a factoring of N. However, as we shall see later [(40) of Sec. 6.6 and (3) of Sec. 7.10 and Sec. 10.5], it is a property of certain extensive homogeneous media that the radiance function N at great distances from the boundaries of such media comes arbitrarily close (for correspondingly great distances) to functions of the form hg, i.e., to factored form, in which there is a spatial factor h and a directional factor g.

The conclusions of the various arguments presented above may now be summarized.

(i) In general media X for which (3) holds, the assumption that there exists a function g on Ξ such that $N(x,\xi) = h(x)\, g(\xi)$ for every x in X and ξ in Ξ is generally untenable on physical grounds (the associated solutions are not unique). More generally, finite representations of the form

$$N(x,\xi) = \sum_{i=0}^{n} h_i(x) g_i(x), \quad n < \infty$$

are also untenable.

(ii) In some extensive, homogeneous media X, there exists a function g on Ξ such that $N(x,\xi) \to h(x)g(\xi)$ for every ξ in Ξ and x sufficiently far from the boundaries of X. By comparing the conclusions summarized in (i) and (ii), we see from (i) that on the one hand the original intuitive guess as to the factorability of N into the form gh was generally incorrect; by conclusion (ii), on the other hand, there is a small solid core of truth inherent in the intuitive guess. Furthermore, while finite representations of N in the form

$$\sum_{i=0}^{n} h_i g_i$$

are generally incorrect, these representations may possibly be so constructed that they increase in accuracy with an increase in the number of terms of the sum. In particular it would seem that by choosing sufficiently large numbers of terms for

$$\sum_{i=0}^{n} h_i g_i \quad ,$$

these approximations to N may be improved at all points of a medium X. Then at large distances from the boundaries of X there will, by (ii), be a single term $h_i g_i$ of

$$\sum_{i=0}^{n} h_i g_i$$

which will dominate the others and which will essentially represent N in those regions.

With these observations we have reached the last stage of the physical motivation for the abstract harmonic representation method. We thereby are led to consider *infinite* series of the form:

$$\sum_{i=0}^{\infty} h_i(x) g_i(\xi)$$

which, for given fixed x in X, represents the radiance distribution values $N(x,\xi)$ for every direction ξ in Ξ.

An Algebraic Setting for Radiance Distributions

The preceding discussion has motivated the representation of a radiance distribution $N(x,\cdot)$ at a fixed point x in an optical medium X by means of an infinite series of functions, in the form:

$$N(x,\xi) = \sum_{i=0}^{\infty} f_i(x) \; \phi_i(\xi) \tag{16}$$

This constitutes the first step in constructing the abstract harmonic representation of $N(x,\cdot)$.

The next step calls for the construction of an infinite family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions, each with Ξ as domain, and with the following properties. First, the ϕ_i's are generally allowed to be complex valued. This provides a great theoretical convenience and in no way forces N to be complex valued under specific physical conditions. Second, we require that the family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be *orthonormal*, i.e.,

$$\int_{\Xi} \phi_i(\xi) \; \overline{\phi}_j(\xi) \; d\Omega(\xi) = \delta_{ij} \tag{17}$$

where δ_{ij} is the Kronecker delta, i.e., δ_{ij} is zero whenever $i \neq j$, and one whenever $i = j$. This operation of integration

and others similar to it will arise sufficiently often in the following discussion that it will be convenient to abbreviate it in general by writing:

$$"[\phi,\psi]" \quad \text{for} \quad \int_{\Xi} \phi(\xi)\,\overline{\psi}(\xi)\,d\Omega(\xi) \tag{18}$$

where ϕ and ψ are any two functions on Ξ so that the integral of their product, as in (18), is defined. The bar over a function denotes complex conjugation. We call $[\phi,\psi]$ the *inner* (or scalar) *product* of ϕ and ψ.

The reason for the terminology "inner product" stems from the deep similarity of this inner product with the classical scalar product $x \cdot y$ of two vectors x and y in euclidean three space. The most striking similarities are paired off in the list below. Their proofs are immediate:

(i)	(i)
If α_1, α_2, α_3 are pairwise orthogonal unit vectors of E_3, then $\alpha_i \cdot \alpha_j = \delta_{ij}$	If $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is an orthonormal family of functions on Ξ, then $[\phi_i,\phi_j] = \delta_{ij}$
(ii)	(ii)
If, for a vector ξ in Ξ there exist three numbers c_1, c_2, c_3 such that $\xi = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$, then $c_i = \xi \cdot \alpha_i$	If, for a function g on Ξ, there exist n numbers c_0, c_1, c_2, ..., c_n, such that $g(\xi) = \sum_{j=0}^{n} c_j\phi_j(\xi)$, then $c_i = [g,\phi_i]$
(iii)	(iii)
$x\cdot(y+z) = x\cdot y + x\cdot z$ $(x+y)\cdot z = x\cdot z + y\cdot z$	$[f,g+h] = [f,g] + [f,h]$ $[f+g,h] = [f,h] + [g,h]$
(iv)	(iv)
$(cx)\cdot y = c(x\cdot y) = x\cdot cy)$	$[cf,g] = c[f,g]$ $[f,cg] = \bar{c}[f,g]$

The physical motivations discussed above have led us to consider *infinite* series, so that the vector-spacelike property (ii) for inner product will be postulated to hold for infinite series. The specific form of the infinite version of (ii) we shall adopt is as follows (the mathematical regularity properties of integrability are omitted for simplicity of exposition):

Completeness property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$. *If F is a function on* Ξ, *and if for every* $j \geq 0$ *we write:*

$$"f_j" \quad \text{for} \quad [F,\phi_j]$$

then:

$$F(\xi) = \sum_{j=0}^{\infty} f_j \, \phi_j(\xi) \tag{19}$$

for every ξ *in* Ξ.

The algebraic setting for radiance distributions discussed in example 15 of Sec. 2.11, now may be used once again. In fact we can easily extend that setting for our present purposes. We therefore imagine all possible radiance distributions at a fixed point x in X and imagine further *all* their negatives and imaginaries ($-N(x,\cdot)$ is the *negative* of $N(x,\cdot)$, $iN(x,\cdot)$ where $i = \sqrt{-1}$, is the *imaginary* of $N(x,\cdot)$) thrown in with them. The totality $\mathcal{N}(x)$ of these and all possible sums of them form a vector space in the general sense: Sums of members of $\mathcal{N}(x)$ are again in $\mathcal{N}(x)$; and multiplication of members of $\mathcal{N}(x)$ by complex numbers are again in $\mathcal{N}(x)$. The additional details of verification are simple and need not detain us here. The main fact to observe is that the set of all integrable radiance distributions at a point x in X can be imbedded in a vector space of functions on Ξ which includes an orthonormal set $\{\phi_0, \phi_1, \phi_2, \ldots\}$ such that the completeness property holds for $\{\phi_0, \phi_1, \phi_2, \ldots\}$. This is the algebraic setting for radiance distributions in which the abstract spherical harmonic method will be discussed.

6.2 Abstract Spherical Harmonic Method

The motivation and prerequisites of the abstract spherical harmonic method having been dispatched in Sec. 6.1, we turn directly to the method itself, now applied to the general time-dependent equation of transfer with source term ((14) of Sec. 3.15):

$$\frac{1}{v}\frac{\partial N}{\partial t} + \xi \cdot \nabla N = -\alpha N + N_* + N_\eta \quad . \tag{1}$$

where N is defined on a general optical medium X which may be finite or infinite, generally inhomogeneous, but isotropic. We assume furthermore that there exists an orthonormal family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions on Ξ which has the completeness property.

The completeness property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ applied to the radiance distribution $N(x,\cdot)$ at x in X yields:

$$N(x,\xi,t) = \sum_{j=0}^{\infty} f_j(x,t) \, \phi_j(\xi) \tag{2}$$

where we have written:

$$\text{"}f_j(x,t)\text{"} \quad \text{for} \quad [N^*(x,\cdot,t), \phi_j] \tag{3}$$

Thus $f_j(x,t)$ is the scalar obtained by performing the integration:

$$\int_{\Xi} N(x,\xi,t)\ \overline{\phi}_j(\xi)\ d\Omega(\xi) \quad .$$

In a similar manner we obtain:

$$N_\eta(x,\xi,t) = \sum_{j=0}^{\infty} f_{\eta,j}(x,t)\ \phi_j(\xi) \tag{4}$$

as the representation of the emission funcion N_η, where we have written:

$$\text{"}f_{\eta,j}(x,t)\text{"} \quad \text{for} \quad [N_\eta(x,\cdot,t),\ \phi_j] \quad . \tag{5}$$

The representation of the volume scattering function σ is next. Since σ uses two directional variables, we use the completeness property twice. First we obtain:

$$\sigma(x;\xi';\xi;t) = \sum_{j=0}^{\infty} \sigma_j(x;\xi';t)\ \phi_j(\xi) \tag{6}$$

where we have written:

$$\text{"}\sigma_j(x;\xi';t)\text{"} \quad \text{for} \quad [\sigma(x;\xi';\cdot;t),\ \phi_j] \tag{7}$$

Next we obtain:

$$\sigma_j(x;\xi';t) = \sum_{k=0}^{\infty} \sigma_{jk}(x,t)\ \overline{\phi}_k(\xi') \tag{8}$$

where we have written:

$$\text{"}\sigma_{jk}(x;t)\text{"} \quad \text{for} \quad [\sigma_j(x;\cdot;t),\ \overline{\phi}_k] \tag{9}$$

Combining these representations, we have:

$$\sigma(x;\xi';\xi;t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sigma_{jk}(x;t)\ \overline{\phi}_k(\xi')\ \phi_j(\xi) \tag{10}$$

The reason for introducing the conjugates of the ϕ_k into (10) will become clear shortly.

Now the whole purpose of the spherical harmonic method, as we have seen in Sec. 6.1, is to effectively separate the spatial variables from the directional variables in the equation of transfer so that the latter may be contained in a system of simple, directly integrable differential equations involving spatial variables only. We now apply the abstract harmonic representations of N, N_η, and σ to the equation of transfer (1), and effect such a separation of variables. On

the right side of (1) we have N_η already represented. Then for the term N* (the summations all go from 0 to ∞):

$$N_*(x,\xi,t) = \int_\Xi \left(\sum_i f_i(x,t)\phi_i(\xi')\right)\left(\sum_j\sum_k \sigma_{jk}(x;t)\bar{\phi}_k(\xi')\phi_j(\xi)\right) d\Omega(\xi')$$

$$= \sum_i f_i(x,t)\left[\int_\Xi \phi_i(\xi')\left(\sum_j\sum_k \sigma_{jk}(x;t)\bar{\phi}_k(\xi')\phi_j(\xi)\right) d\Omega(\xi')\right]$$

$$= \sum_i f_i(x,t)\left[\sum_j\sum_k \sigma_{jk}(x;t)\phi_j(\xi)\int_\Xi \phi_i(\xi')\bar{\phi}_k(\xi')\, d\Omega(\xi')\right]$$

$$= \sum_i f_i(x,t)\left[\sum_j\sum_k \sigma_{jk}(x;t)\phi_j(\xi)\,\delta_{ik}\right]$$

$$= \sum_i f_i(x,t)\left[\sum_j \sigma_{ji}(x;t)\phi_j(\xi)\right]$$

$$= \sum_j \left[\sum_i f_i(x,t)\sigma_{ji}(x;t)\right]\phi_j(\xi) \tag{11}$$

Since the medium X is assumed isotropic, the volume attenuation function values $\alpha(x;\xi)$ are independent of ξ, and so α need not be represented by a series of the complete family $\{\phi_0, \phi_1, \phi_2, \ldots\}$. By means of (4), (10), and (11) we can therefore represent the right side of (1) in the form:

$$\sum_{j=0}^{\infty}\left[-\alpha(x)f_j(x,t) + \sum_{i=0}^{\infty} f_i(x,t)\sigma_{ji}(x;t) + f_{\eta,j}(x,t)\right]\phi_j(\xi) \tag{12}$$

Attention is now directed to the left side of (1). The time derivative term is directly treated to yield:

$$\sum_{j=0}^{\infty} \frac{1}{v}\frac{\partial f_j(x,t)}{\partial t}\phi_j(\xi) \quad . \tag{13}$$

The spatial derivative term becomes:

$$\xi \cdot \nabla N(x,\xi,t) = \xi \cdot \nabla\left(\sum_{j=0}^{\infty} f_j(x,t)\ \phi_j(\xi)\right)$$

$$= \sum_{j=0}^{\infty}\left[\xi \cdot \nabla f_j(x,t)\right]\phi_j(\xi) \qquad (14)$$

Combining (12), (13), and (14) according to (1), we have:

$$\sum_{j=0}\left[\frac{1}{v}\ \frac{\partial f_j(x,t)}{\partial t} + \xi \cdot \nabla f_j(x,t) + \alpha(x)\ f_j(x,t) - \sum_{i=0} f_i(x,t)\sigma_{ji}(x;t) - f_{\eta,j}(x,t)\right]\ \phi_j(\xi) = 0 \qquad (15)$$

If it weren't for the spatial derivative term the contents of the square bracket would have been free of the variable ξ, and a system of equations would have been obtained by setting each bracketed jth term to zero. At any rate we can eliminate the presence of ξ by an integration over Ξ. The orthonormality property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is available for use in this task. Thus multiplying each side of (15) by $\overline{\phi}_k(\xi)$ and integrating over Ξ, the orthonormality property immediately yields

$$\frac{1}{v}\ \frac{\partial f_k(x,t)}{\partial t} + \sum_{j=0}^{\infty}\int_{\Xi} \xi \cdot \nabla f_j(x,t)\ \phi_j(\xi)\ \overline{\phi}_k(\xi)\ d\Omega(\xi)$$

$$= -\ \alpha(x) f_k(x,t) + \sum_{j=0}^{\infty} f_i(x;t)\ \sigma_{jk}(x;t) + f_{\eta,k}(x,t) \qquad (16)$$

If we now write:

$$"D_{jk}" \quad \text{for} \quad \int_{\Xi} \xi \cdot \nabla(\ )\ \phi_j(\xi)\ \overline{\phi}_k(\xi)\ d\Omega(\xi) \quad , \qquad (17)$$

then we obtain, at last, the spherical harmonic analysis of (1):

$$\boxed{\frac{1}{v} \cdot \frac{\partial f_k}{\partial t} + \sum_{j=0}^{\infty} f_j D_{jk} = -\ \alpha\ f_k + \sum_{j=0}^{\infty} f_j\ \sigma_{jk} + f_{\eta,k} \qquad k = 0, 1, 2, \ldots} \qquad (18)$$

This is the requisite abstract spherical harmonic system of partial differential equations for the family $\{f_0, f_1, f_2, \ldots\}$ of functions, the *abstract harmonic coefficient functions* of the radiance distribution $N(x,\cdot)$. Knowledge of these f_j

allows construction of $N(x,\cdot)$ according to (2). The heart of the abstract harmonic method of solving the equation of transfer thus resides in (18).

Finite Forms of the Abstract Harmonic Equations

An inspection of the system (18) of abstract harmonic equations governing the harmonic coefficient functions f_k shows two infinite series involved in the system. The presence of these infinite series could occasionally negate the practical utility of the system, for example in numerical solution work. It is interesting to observe, however, that these infinite series may be rigorously removed and replaced by finite sums under the combined action of two very general conditions, one physical, the other mathematical. The mathematical condition simplifies the differential operator series; the physical condition simplifies the scattering term series. We shall now briefly indicate the nature of these conditions.

We shall say that the family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions on Ξ has the *finite recurrence property* of degree ν if for every element ξ' in Ξ and every ϕ_j in the family, there exist ν constants A_{jk} and ν elements $\phi_{\alpha_1}, \ldots, \phi_{\alpha_\nu}$ of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ such that

$$\xi \cdot \xi' \phi_j(\xi) = \sum_{k=1}^{\nu} A_{jk}\, \phi_{\alpha_k}(\xi) \tag{19}$$

holds for every ξ in Ξ. The motivation for this property arises in an attempt to simplify the form of the operators D_{jk} and to reduce to a finite series the infinite series involving them in (18). For example, in an orthogonal, three-dimensional coordinate frame in which $x = (x_1, x_2, x_3)$, we have:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}$$

We use this form in (17) to obtain the representation

$$D_{jk} = a_{jk}\frac{\partial}{\partial x_1} + b_{jk}\frac{\partial}{\partial x_2} + c_{jk}\frac{\partial}{\partial x_3} \tag{20}$$

where we have written:

$$\text{"}a_{jk}\text{" for } \int_\Xi \xi \cdot \mathbf{i}\, \phi_j(\xi)\, \overline{\phi}_k(\xi)\, d\Omega(\xi) \tag{21}$$

$$\text{"}b_{jk}\text{" for } \int_\Xi \xi \cdot \mathbf{j}\, \phi_j(\xi)\, \overline{\phi}_k(\xi)\, d\Omega(\xi) \tag{22}$$

$$"c_{jk}" \quad \text{for} \quad \int_{\Xi} \xi \cdot \mathbf{k}\, \phi_j(\xi)\, \overline{\phi}_k(\xi)\, d\Omega(\xi) \tag{23}$$

By postulating a finite recurrence property of degree ν for $\{\phi_0, \phi_1, \phi_2, \ldots\}$, if follows that $a_{jk} = 0$ whenever the indices k and j differ by a sufficiently large amount: indeed $a_{jk} = 0$ for all but at most ν terms. Similarly with b_{jk} and c_{jk}. This means that for fixed k $D_{jk} = 0$ whenever j is sufficiently large, and so the number of terms on the left of (18) become finite in the present case. It turns out that any orthonormal family obtained from suitable nth order ordinary differential equations (a rich source of orthonormal families by means of Sturm-Liouville theory) will possess a finite recurrence property of degree ν.

Finally, the physical condition which simplifies the abstract harmonic equations is that of isotropy of the medium. In the present case the isotropy reduces the general functional dependence of σ on the independent variables ξ' and ξ to the special dependence of σ on the scalar product $\xi' \cdot \xi$ of the directions. This simplified structure of σ in turn manifests itself in a simplification of the representation (10) to the form:

$$\sigma(x;\xi';\xi;t) = \sum_{j=0}^{\infty} \sigma_j(x;t)\, \overline{\phi}_j(\xi')\, \phi_j(\xi) \quad . \tag{24}$$

We shall not go into the derivation details of this relation in the present abstract case. It suffices to note that this form can be obtained when the members of the orthonormal family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ obey a general type of *addition theorem* often valid for functions arising in Sturm-Liouville theory. Examples of addition theorems for such functions, are, e.g., in [318]. (See (12) and (15) of Sec. 6.3.)

The simplifying effect of (24) becomes evident when we recalculate $N_*(x,\xi,t)$ after the manner of (11):

$$N_*(x,\xi,t) = \int_{\Xi} \left(\sum_i f_i(x,t)\, \phi_i(\xi')\right)\left(\sum_j \sigma_j(x;t)\overline{\phi}_j(\xi')\, \phi_j(\xi)\right) d\Omega(\xi')$$

$$= \sum_i f_i(x,t) \left[\int_{\Xi} \phi_i(\xi')\left(\sum_j \phi_j(x;t)\overline{\phi}_j(\xi')\phi_j(\xi)\right)\right] d\Omega(\xi')$$

$$= \sum_i f_i(x,t)\left[\sum_j \sigma_j(x;t)\phi_j(\xi) \int_{\Xi} \phi_i(\xi')\overline{\phi}_j(\xi')\right] d\Omega(\xi')$$

$$= \sum_i f_i(x,t) \left[\sum_j \sigma_j(x;t)\ \phi_j(\xi)\ \delta_{ij} \right]$$

$$= \sum_i f_i(x,t)\ \sigma_i(x;t)\ \phi_i(\xi) \qquad (25)$$

By combining the preceding two conditions, the total effect on (18) is a complete finitization of each equation in the system of equations, thereby rendering them more effective for numerical computations. We may summarize these constructions as follows:

Let X be an arbitrary isotropic, inhomogeneous optical medium with internal emission radiance function N_η and general time-dependent radiance field N as governed by the equation of transfer (1). Let $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be an orthonormal family of functions defined on the unit sphere Ξ such that: the family (a) possesses the completeness property (see (19) of Sec. 6.1); (b) possesses the finite recurrence property (19); (c) satisfies an addition theorem (24). Then each member of the general abstract harmonic system of partial differential equations (18) reduces to the following finite form: For some positive integer ν:

$$\boxed{\frac{1}{v}\frac{\partial f_k}{\partial t} + \sum_{j=0}^{\nu} f_j D_{jk} = -\alpha f_k + f_k \sigma_k + f_{\eta,k} \qquad k = 0,1,2,\ldots}$$

$$(26)$$

6.3 Classical Spherical Harmonic Method: General Media

The general theory of the abstract harmonic method developed in the preceding section will now be illustrated for the classical case in which the orthonormal family is constructed from familes of associated Legendre functions of the first kind and circular (trigonometric) functions. The optical medium X will be generally inhomogeneous and isotropic, with time varying inherent optical properties, and given internal sources.

The Orthonormal Family

We begin by observing that the classical spherical harmonic method customarily uses the ordered pair (μ,ϕ) of numbers to specify a point ξ in Ξ, where we have written "μ" for cos θ, and where (θ,ϕ) are the two angles customarily used to specify ξ in Ξ (see Sec. 2.4 and also example 14 of Sec. 2.11 for an earlier use of μ in conjunction with Legendre polynomials). The range of the variable μ is thus the interval [-1,1], and the range of ϕ, [0,2π]. Every ξ in Ξ determines a unique (θ,ϕ), that is a unique μ in [-1,1] and a unique ϕ in [0,2π].

Conversely, any pair (μ,ϕ) in $[-1,1] \times [0,2\pi]$ determines a unique ξ in Ξ.

The values of associated Legendre functions are usually denoted by "$P_n^m(\mu)$". The integer n is nonnegative, i.e., $n \geq 0$ and the integer m satisfies the inequalities: $-n \leq m \leq n$. The general relations in the theory of Legendre polynomials we shall use below may be found fully developed, e.g., in [318], [289], and [119]. In particular we shall note that:

$$P_n^{-m} = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m \tag{1}$$

and that:

$$\left.\begin{aligned} P_n^o &= P_n \\ P_n^m &= 0 \quad \text{for} \quad m > n \end{aligned}\right\} \tag{2}$$

where "P_n" denotes the Legendre function of the first kind and of degree n. For our present purposes, we note that the associated Legendre function P_n^m is a real valued function with domain $[-1,1]$ and defined for all integers m,n such that n is nonnegative and $|m| \leq n$. The associated Legendre functions include, by (2), the Legendre polynomials as special cases. Any functions P_n^m arising in the subsequent discussions for which $n < 0$, are to be zero-valued functions. In view of (1) and (2) only P_n^m with n+1 nonnegative indices m need be tabulated.

The orthogonality property of the family of associated Legendre functions takes the form:

$$\int_{-1}^{1} P_n^m(\mu)\, P_r^m(\mu)\, d\mu = \begin{cases} 0, & , \text{ whenever } n \neq r \\ \dfrac{2}{2n+1} \cdot \dfrac{(n+m)!}{(n-m)!}, & \text{whenever } n = r \end{cases} \tag{3}$$

The integral properties of the family of circular functions needed here are summarized by the equations:

$$\begin{aligned} &\int_0^{2\pi} \sin m\phi \, d\phi = 0 \\ &\int_0^{2\pi} \cos m\phi \, d\phi = \begin{cases} 0 & \text{if } m \neq 0 \\ 2\pi & \text{if } m = 0 \end{cases} \end{aligned} \tag{4}$$

where m is confined to integral values. These properties can be succinctly summarized by using complex variables. Thus, all three equations in (4) may be expressed by writing:

$$\int_0^{2\pi} e^{im\phi}\, d\phi = 2\pi\delta_{om} \tag{5}$$

where δ_{om} is an instance of the general Knonecker delta δ_{ij}. The use of complex variables will considerably facilitate our work in this section, and so they will be retained throughout. One can always return to the real number setting by finding and considering separately the real and imaginary parts of a complex term.

The details of the construction of the requisite orthonormal family on Ξ are clearly indicated by considering (3) and (5). Thus to an arbitrary ξ in Ξ, (to which corresponds a unique pair (μ,ϕ)) and to every pair of integers m, n, with $n \geq 0$, $|m| \leq n$ we assign the complex number $\Phi_n^m(\xi)$ where we have written:

$$\text{"}\Phi_n^m(\xi)\text{"} \quad \text{for} \quad A_n^m\, P_n^m(\mu)\, e^{im\phi} \tag{6}$$

where in turn we have written

$$\text{"}A_n^m\text{"} \quad \text{for} \quad \left[\frac{(2n+1)\,(n-m)!}{4\pi(n+m)!}\right]^{1/2} \tag{7}$$

By observing that:

$$A_n^{-m} = \frac{(n+m)!}{(n-m)!}\, A_n^m$$

we can limit tabulations of A_n^m to nonnegative indices m. Furthermore, by recalling (1), the complex conjugate of $\Phi_n^m(\xi)$ may be expressed as follows:

$$\overline{\Phi_n^m(\xi)} = A_n^m\, P_n^m(\xi)\, e^{-im\phi} = (-1)^m\, \Phi_n^{-m}(\xi) \tag{8}$$

The orthonormality property of the family of functions Φ_n^m over Ξ may now be verified. For example:

$$\int_\Xi \Phi_n^m(\xi)\, \overline{\Phi_r^m(\xi)}\, d\Omega(\xi) = A_n^m\, A_r^m \quad 2\pi \int_{-1}^{+1} P_n^m(\mu)\, P_r^m(\mu)\, d\mu$$

$$= 2\pi A_n^m\, A_r^m\, \delta_{nr}\, \frac{2}{(2n+1)} \cdot \frac{(n+m)!}{(n-m)!}$$

$$= \delta_{nr}$$

The remaining case where the upper indices of Φ_n^m may differ is straightforward using (5). Hence we have:

$$\int_{\Xi} \Phi_n^m(\xi)\ \Phi_a^b(\xi)\ d\Omega(\xi) = \delta_{mb}\ \delta_{na} \tag{9}$$

for every n, a $\geq$ 0 and b, m such that $|b| \leq a$, $|m| \leq n$.

An exact one-to-one correspondence can be established between the abstract family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of Sec. 6.2 and the spherical harmonic family presently under consideration. Thus to ϕ_j of the earlier discussion we pair Φ_n^m, where $j = n^2 + m + n$. This correspondence arises when one contemplates Fig. 6.1 in which each dot in the figure is paired with the integer couple (m,n), $n \geq 0$, $|m| \leq n$, corresponding to the indices of Φ_n^m. Then counting each row of dots by reading from left to right and counting rows from bottom to top, each dot is given a single index j. For example the dot in the first row, corresponding to (0,0) is given the index 0. The dot corresponding to (-1,1) is given the index 1, (0,1) the index 2, (-3,4) the index 17, etc. In general:

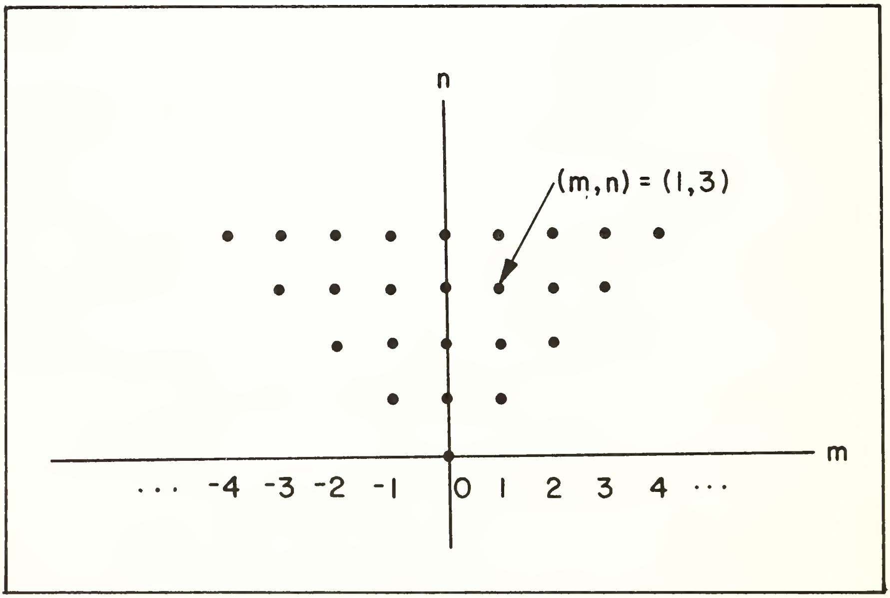

FIG. 6.1 Scheme for establishing the correspondence between the abstract and classical spherical harmonic method.

$$(m,n) \text{ is paired with the index } j = n^2 + m + n \tag{10}$$

and

$$\Phi_n^m \text{ is paired with } \phi_j \tag{11}$$

Observe that the pairings are unique: given (m,n) there is precisely one $j > 0$ corresponding to this pair; given $j > 0$, there is precisely one pair (m,n) on the array corresponding to j and is readily obtained under the conditions on m,n described above.

Properties of the Orthonormal Family

We shall now show that the family of spherical harmonics Φ_n^m on Ξ possesses the three main properties sufficient to insure a reduction of the general abstract harmonic system (18) of Sec. 6.2 to its finite version (26) of Sec. 6.2. (The proof of the orthonormality of the family of spherical harmonics was outlined in the discussion leading to (9).)

The *completeness property* of the set of spherical harmonics holds. However, the property depends on some relatively advanced arguments, and the interested reader is referred to Chapter 7 of [47] for the general theory of completeness of families of functions arising from nth order differential equations.

The *addition theorem* for Legendre functions holds and takes the form (see, e.g., [119]):

$$P_n(\xi\cdot\xi') = P_n(\mu)\, P_n(\mu') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n(\mu)\, P_n(\mu') \cos m(\phi-\phi') \tag{12}$$

where ξ and ξ' are any two directions in Ξ and (μ,ϕ), (μ',ϕ') are their corresponding angular representations. Using (1), (2), the evenness of cosine, and the oddness of sine, (12) may be compactly written as:

$$P_n(\xi\cdot\xi') = \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\mu)\, P_n^m(\mu')\, e^{im(\phi-\phi')} \tag{13}$$

The argument of P_n in (13) is the scalar product of ξ' and ξ. This scalar product is reminiscent of the isotropy condition for an optical medium. We now show how the isotropy condition leads in the present case to the representation of σ in the form of (24) of Sec. 6.2. When isotropy

holds, the value of σ (for a fixed x and t) is known once $\xi \cdot \xi'$ is known, i.e., once a number $\mu = \xi \cdot \xi'$ in the interval $[-1,1]$ is specified. This value of σ under isotropy conditions will be denoted by "$\sigma(x;\xi\cdot\xi';t)$". Therefore, the family of Legendre polynomials P_n being complete (a fact also supplied by the general theory in [47] cited above), we may express $\sigma(x;\xi\cdot\xi;t)$ as follows:

$$\sigma(x;\xi\cdot\xi';t) = \frac{1}{2\pi}\sum_{j=0}^{\infty}\left[\frac{2j+1}{2}\right]\sigma_j(x;t)\,P_j(\xi\cdot\xi') \tag{14}$$

where we have written:

$$\text{"}\sigma_j(x;t)\text{" for } 2\pi\int_{-1}^{1}\sigma(x;\mu;t)\,P_j(\mu)\,d\mu \tag{15}$$

Using (13) to represent $P_j(\xi\cdot\xi')$ in (14), we have:

$$\begin{aligned}\sigma(x;\xi\cdot\xi';t) &= \frac{1}{2\pi}\sum_{j=0}^{\infty}\left[\frac{2j+1}{2}\right]\sigma_j(x;t)\sum_{m=-j}^{j}\frac{(j-m)!}{(j+m)!}P_j^m(\mu)\,P_j^m(\mu')\,e^{im(\phi-\phi')}\\ &= \sum_{j=0}^{\infty}\sigma_j(x;t)\sum_{m=-j}^{j}\overline{\Phi_j^m(\xi')}\;\Phi_j^m(\xi)\end{aligned} \tag{16}$$

This is reducible to the form of (24) of Sec. 6.2 as may be seen by using the correspondence between ϕ_j and Φ_n^m established above. (To show the correspondence in complete detail, let $\sigma_j(x;t)$ be denoted *ad hoc* as "$\sigma_j^m(x;t)$" and require it to have value $\sigma_j(x;t)$ for m in the range $-j \leq m \leq j$.)

In this way we see how the addition theorem for the P_n^m and the isotropy condition on scattering combine to form the extremely useful representation (16). The reader may now extend this idea to still other complete orthonormal families of functions defined on $[-1,1]$ provided an addition theorem of the kind (13) is available for the family.

Next, we observe that the orthonormal family of functions Φ_n^m satisfies the *finite recurrence property* of degree 2. This observation is based on the following three well-known

recurrence properties of associated Legendre functions (see, e.g., [289], [119]):

$$\mu P_n^m(\mu) = \frac{(n+m)\ P_{n-1}^m(\mu) + (n+1-m)\ P_{n+1}^m(\mu)}{2n+1} \tag{17}$$

$$\sin\theta\ P_n^m(\mu) = \frac{P_{n+1}^{m+1}(\mu) - P_{n-1}^{m+1}(\mu)}{(2n+1)} \tag{18}$$

$$\sin\theta\ P_n^m(\mu) = \frac{(n-m+2)(m-n-1)\ P_{n+1}^{m-1}(\mu) + (n+m-1)(n+m)\ P_{n-1}^{m-1}(\mu)}{(2n+1)} \tag{19}$$

As an example of how these recurrence relations give rise to instances of the general recurrence property (19) of Sec. 6.2, consider (17). Here we recall that "μ" denotes $\xi\cdot\mathbf{k}$; $\mathbf{k}$ is the unit vector along the positive z-axis. Hence ξ' in (19) of Sec. 6.2 is now $\mathbf{k}$. Next, multiply each side of (17) by $A_n^m\ e^{im\phi}$. Applying the general definition (6) and making some algebraic rearrangements, the net result is:

$$\xi\cdot\mathbf{k}\ \Phi_n^m(\xi) = C(n,m)\ \Phi_{n-1}^m(\xi) + C(n+1,m)\ \Phi_{n+1}^m(\xi) \tag{20}$$

where we have written:

$$\text{"}C(n,m)\text{"} \quad \text{for} \quad \left[\frac{(n-m)(n+m)}{(2n-1)(2n+1)}\right]^{1/2} \tag{21}$$

Hence in (19) of Sec. 6.2, we have $\nu = 2$, and the A_{jk} are now in the form of $C(j,k)$, with $j = n^2 + m + n$, and $\alpha_1 = (n-1)^2 + m + (n-1)$, $\alpha_2 = (n+1)^2 + m + (n+1)$. The specific representation of $\xi\cdot\mathbf{k}\ \Phi_n^m(\xi)$ in (20) is now used in (20) of Sec. 6.2 to effect an evaluation of the number c_{jk}, and hence the sum:

$$\sum_{j=0}^{\nu} c_{jk} \frac{\partial f_j}{\partial x_3} \tag{22}$$

which forms part of the operation:

$$\sum_{j=0}^{\nu} f_j D_{jk} \tag{23}$$

in (26) of Sec. 6.2. To see how (22) is evaluated, let us represent $N(x,\xi,t)$ by means of the functions Φ_n^m:

$$N(x,\xi,t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_n^m(x,t)\ \Phi_n^m(\xi) \tag{24}$$

where we have written:

$$"F_n^m(x,t)" \quad \text{for} \quad \int_{\Xi} N(x,\xi,t)\ \overline{\Phi_n^m(\xi)}\ d\Omega(\xi) \tag{25}$$

Thus F_n^m in the present context corresponds to f_j in the abstract context of Sec. 6.2, just as Φ_n^m corresponds to ϕ_j. Furthermore, the correspondence of j in f_j with the pair of indices (m,n) of F_n^m is once again that established above. (See Fig. 6.1 and (10), (11).)

Returning to (22), we consider it in the context of (18) of Sec. 6.2, but now using the present family $\{\Phi_n^m\}$ of orthonormal functions. We threfore are to consider:

$$\sum_{j=0}^{\infty} c_{jk} \frac{\partial f_j}{\partial x_3} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ \int_{\Xi} \xi\cdot k\,\Phi_n^m(\xi)\,\overline{\Phi_a^b(\xi)}\,d\Omega(\xi) \right\} \frac{\partial F_n^m}{\partial x_3}$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ \int_{\Xi} \left[C(n,m)\,\Phi_{n-1}^m(\xi) + C(n+1,m)\ \Phi_{n+1}^m(\xi) \right] \Phi_a^b(\xi)d\Omega(\xi) \right\} \frac{\partial F_n^m}{\partial x_3}$$

$$= C(a+1,b)\ \frac{\partial F_{a+1}^b}{\partial x_3} + C(a,b)\ \frac{\partial F_{a-1}^b}{\partial x_3} \tag{26}$$

in which $k = a^2 + b + a$.

Thus the infinite sum of z-derivatives in (18) of Sec. 6.2 is reduced to a sum of two such derivatives.

The general procedure should now be clear: by placing the recurrence relations (18) and (19) into their appropriate counterparts of (20), the numbers a_{jk} and b_{jk} in (21), (22) of Sec. 6.2 are readily evaluated. Then the sums:

$$\sum_{j=0}^{\nu} a_{jk} \frac{\partial f_j}{\partial x_1} \quad , \quad \sum_{j=0}^{\nu} b_{jk} \frac{\partial f_j}{\partial x_2}$$

are evaluated analogously to the manner displayed in (26). These details may be left to the reader.

General Equations for Spherical Harmonic Method

The net result of the reduction calculations on (26) outlined above may be written in the form:

$$\boxed{\begin{aligned} &\frac{1}{v}\frac{\partial F_a^b(x,t)}{\partial t} + \left[C(a,b)\frac{\partial F_{a-1}^b}{\partial x_3} + C(a+1,b)\frac{\partial F_{a+1}^b}{\partial x_3}\right] \\ &\quad + \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\left[B(a,b)F_{a-1}^{b-1}(x,t) - B(a+1,-b+1)\ F_{a+1}^{b-1}(x,t)\right] \\ &\quad + \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\left[-B(a,-b)F_{a-1}^{b+1}(x,t) + B(a+1,b+1)F_{a+1}^{b+1}(x,t)\right] \\ &\quad = \left[-\alpha(x,t) + \sigma_a(x;t)\right]F_a^b(x,t) + F_{\eta,a}^n(a,t) \\ &\quad a = 0,1,2,\ldots;\ |b| \le a. \end{aligned}}$$

(27)

where we have written:

$$\text{"}B(a,b)\text{" for } \left[\frac{(a+b)(a+b-1)}{(2a-1)(2a+1)}\right]^{1/2} \tag{28}$$

and where $C(a,b)$ is defined generally in (21). Furthermore, we have written:

$$\text{"}F_{\eta,a}^b(x,t)\text{" for } \int_\Xi N_\eta(x,\xi,t)\ \overline{\Phi_a^b(\xi)}\ d\Omega(\xi) \tag{29}$$

analogously to (25), so that N_η has the representation:

$$N_\eta(x,\xi,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} F_{\eta,n}^m(x,t)\ \Phi_n^m(\xi) \quad . \tag{30}$$

The set of equations (27) forms a coupled infinite system of equations in the unknown functions F_a^b, $a = 0,1,2,\ldots$, $|b| \le a$. The functions F_a^b are generally complex valued, according to their defined construction (25), and such that $N(x,\xi,t)$ is real valued, according to (24). The general initial conditions for the system (27) are:

$$F_a^b(x,0) = \int_{\Xi} N^o(x,\xi,0)\ \overline{\Phi_a^b(\xi)}\ d\Omega(\xi) \quad , \tag{31}$$

for every x in X, and where N^o is the given initial radiance function on $X \times \Xi$ at $t = 0$. For steady state versions of (27), the time derivative term is zero. The functions F_a^b then have domain X and (31) is replaced by:

$$F_a^b(x_o) = \int_{\Xi} N^o(x_o,\xi)\ \overline{\Phi_a^b(\xi)}\ d\Omega(\xi) \tag{32}$$

for x_o over some appropriate subset of the boundary of X (cf., e.g., (26) of Sec. 6.4).

The system (27) is of sufficient generality to solve such problems as point source, beam source, and general internal source problems in the sea; natural light field problems in lakes, harbors, and the sea. Observe that the inherent optical properties in the form of α and σ_a may be quite general, and that the term $F_{\eta,a}^b$ provides for internal sources of radiant flux, such as artificial light sources (laser beams, searchlights, submerged incandescent point sources, etc.) or natural light sources (phosphorescence, animal sources, etc.). The general methods of solution of (27) and its manifold variants are well known and may be implemented by programmed machine procedures. If the model is sufficiently simple (as, e.g., in the illustration of Sec. 6.4) the associated simplified form of system (27) may be solved by hand and evaluated numerically or even used for general theoretical reasoning.

6.4 Classical Spherical Harmonic Method: Plane-Parallel Media

The classical spherical harmonic method developed in the preceding section for general media will now be illustrated in a setting of primary importance in hydrologic (and meteorologic) optics: the plane-parallel optical medium. Throughout this section, then, we shall assume that X is a plane-parallel medium of arbitrary (finite or infinite) depth. The incident light field and the optical properties of X are assumed to be in the steady state and independent of the x and y coordinates throughout X, thus establishing a stratified medium and stratified steady radiance field throughout X.

Under the present conditions on the medium X, the general system of equations (27) of Sec. 6.3 reduces to:

$$\boxed{C(a,b)\,\frac{\partial F_{a-1}^b}{-\partial z} + C(a+1,b)\,\frac{\partial F_{a+1}^b}{-\partial z} = (-\alpha + \sigma_a)\,F_a^b + F_{\eta,a}^b \qquad a = 0,1,2,\ldots;\ |b| \le a} \tag{1}$$

Here we have adopted the terrestrially based coordinate system for hydrologic optics (Sec. 2.4) wherein depth z is measured positive downwards from the air-water boundary. Thus "$-z$" in (1) now replaces "x_3" in the general formula (27) of Sec. 6.3, and "x" and "y" replace "x_1" and "x_2", respectively. The functions α and σ_a may vary with depth.

The first few equations of system (1), written out in groups for each value of a, are:

a = 0; b = 0:

$$C(1,0)\,\frac{\partial F_1^0}{-\partial z} = (-\alpha+\sigma_o)\;F_0^0 + F_{\eta,0}^0$$

a = 1; b = -1:

$$C(a,-1)\,\frac{\mu F_2^{-1}}{-\partial z} = (-\alpha+\sigma_1)\;F_1^{-1} + F_{\eta,1}^{-1}$$

a = 1; b = 0:

$$C(1,0)\,\frac{\partial F_o^o}{-\partial z} + C(2,0)\,\frac{\partial F_2^0}{-\partial z} = (-\alpha+\sigma_1)\;F_1^0 + F_{\eta,1}^0$$

a = 1; b = 1:

$$C(2,1)\,\frac{\partial F_2^1}{-\partial z} = (-\alpha+\sigma_1)\;F_1^1 + F_{\eta,1}^1$$

a = 2; b = -2:

$$C(3,2)\,\frac{\partial F_3^{-2}}{-\partial z} = (-\alpha+\sigma_2)\;F_2^{-2} + F_{\eta,2}^{-2}$$

a = 2; b = -1:

$$C(2,-1)\,\frac{\partial F_1^{-1}}{-\partial z} + C(3,-1)\,\frac{\partial F_3^{-1}}{-\partial z} = (-\alpha+\sigma_2)\;F_2^{-1} + F_{\eta,2}^{-1}$$

a = 2; b = 0:

$$C(2,0)\,\frac{\partial F_1^0}{-\partial z} + C(3,0)\,\frac{\partial F_3^0}{-\partial z} = (-\alpha+\sigma_2)\;F_2^0 + F_{\eta,2}^0$$

a = 2; b = 1:

$$C(2,1)\,\frac{\partial F_1^1}{-\partial z} + C(3,1)\,\frac{\partial F_3^1}{-\partial z} = (-\alpha+\sigma_2)\;F_2^1 + F_{\eta,2}^1$$

a = 2; b =2:

$$C(3,2)\,\frac{\partial F_3^2}{-\partial z} = (-\alpha+\sigma_2)\;F_2^2 + F_{\eta,2}^2$$

Thus the group of equations for $a = 0$ consists of one equation; the group for $a = 1$ consists of three equations; the group for $a = 2$ consists of five equations. In general the group for $a = n$ consists of $2n + 1$ equations. Some of the derivative terms are missing in the displayed system above because of the conditions placed on the indices at the outset of the discussion. Thus $F_a^b = 0$ if $a < 0$ or $a < |b|$. A convenient auxiliary rule to observe in this respect is that: whenever $a-b = 0$ or $a+b = 0$, then $C(a,b) = 0$.

A Formal Solution Procedure

The system (1), which represents the system of equations for the spherical harmonic method in a plane-parallel setting, displays an interesting type of coupling among the functions F_a^b. Observe how the upper index b is fixed in each equation of the system. We shall now show how this feature permits a simplification of the general solution procedure of the system. The manner of simplification may be easily seen by means of the diagram in Fig. 6.2.

Each dot in Fig. 6.2 represents an ordered pair (b,a) of indices corresponding to F_a^b. The effect of the rather weak coupling among the unknown functions F_a^b of system (1) is such that we can partition the set of unknown functions into subsets, corresponding to the vertical columns of dots, and

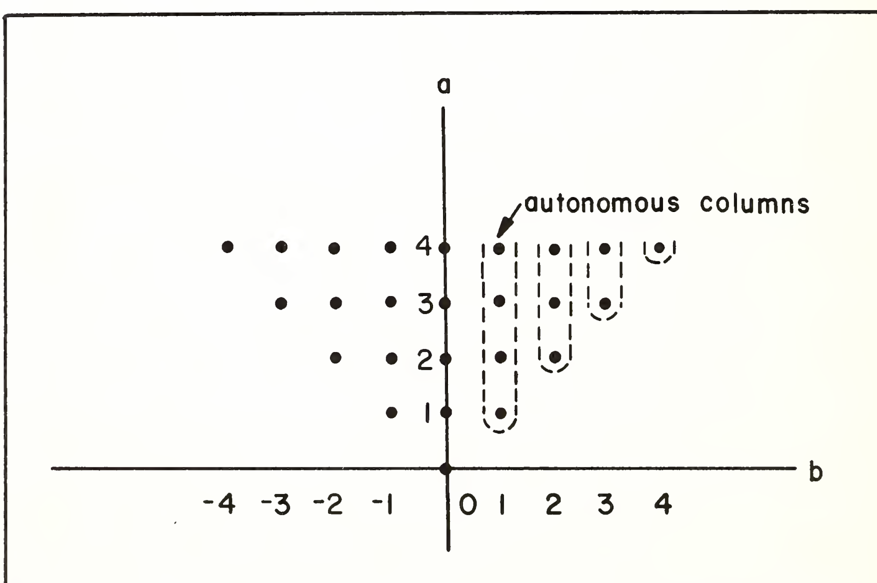

FIG. 6.2 A way of grouping the functions F_a^b into autonomous families, for solution purposes.

solve for the unknown functions associated solely with each column. That is, the unknowns F_a^b in the b^{th} column can be obtained independently of the unknowns in the other columns of of the array. This observation can be put into a convenient mathematical form as follows. Let us write:

$$\text{"}\mathbf{F}^b\text{"} \quad \text{for} \quad (F^b_{|b|}\ F^b_{|b|+1},\ F^b_{|b|+2},\ \ldots) \tag{2}$$

and

$$\text{"}\mathbf{F}^b_\eta\text{"} \quad \text{for} \quad \left(F^b_{\eta,|b|},\ F^b_{\eta,|b|+1},\ F^b_{\eta,|b|+2},\ \ldots\right) \tag{3}$$

Thus, e.g.,

$$\mathbf{F}^0 = (F_0^0,\ F_1^0,\ F_2^0,\ \ldots)$$

$$\mathbf{F}^1 = (F_1^1,\ F_2^1,\ F_3^1,\ \ldots)\ , \qquad \mathbf{F}^{-1} = (F_1^{-1},\ F_2^{-1},\ F_3^{-1},\ \ldots)\ ,$$

$$\mathbf{F}^2 = (F_2^2,\ F_3^2,\ F_4^2,\ \ldots)\ , \qquad \mathbf{F}^{-2} = (F_2^{-2},\ F_2^{-2},\ F_4^{-2},\ \ldots)\ ,$$

and so on. With this notation, we see that the part of system (1) corresponding to an arbitrary fixed index b may be written succinctly in vector form as:

$$-\frac{d}{dz}\,(\mathbf{F}^b\,\mathcal{C}^b) = \mathbf{F}^b\,\mathcal{A} + \mathbf{F}^b_\eta \tag{4}$$

where we have written:

$$\text{"}\mathcal{C}^b\text{"} \quad \text{for} \quad \begin{bmatrix} 0 & C(|b|+1,b) & 0 & 0 & 0 \ldots \\ C(|b|+1,b) & 0 & C(|b|+2,b) & 0 & 0 \ldots \\ 0 & C(|b|+2,b) & 0 & C(|b|+3,b) & 0 \ldots \\ 0 & 0 & C(|b|+3,\ b) & 0 & C(|b|+3,b) \ldots \\ 0 & 0 & 0 & C(|b|+4,b) & 0 \ldots \\ 0 & 0 & 0 & 0 & C(|b|+5,b) \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \tag{5}$$

and where we have written:

$$"\mathcal{A}" \text{ for } \begin{bmatrix} -\alpha+\sigma_0 & 0 & 0 & 0 & \cdots \\ 0 & -\alpha+\sigma_1 & 0 & 0 & \cdots \\ 0 & 0 & -\alpha+\sigma_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (6)$$

The system (4) may be rearranged into the form:

$$\boxed{\begin{array}{l} -\dfrac{d\mathbf{F}^b}{dz} = \mathbf{F}^b \mathcal{B}^b + \mathbf{G}^b_\eta \\ b = 0, \pm 1, \pm 2, \ldots \end{array}} \quad (7)$$

where we have written:

$$"\mathcal{B}^b" \text{ for } \mathcal{A}(\mathcal{C}^b)^{-1} \quad (8)$$

$$"\mathbf{G}^b_\eta" \text{ for } \mathbf{F}^b_\eta(\mathcal{C}^b)^{-1} \quad (9)$$

and where "$(\mathcal{C}^b)^{-1}$" denotes the formal inverse of $\mathcal{C}^b$.

The formal solution procedure for (1) is now seen to be reduced to that associated with (7) and thereby becomes relatively straightfoward on either the numerical or manual levels. Of course, in practice, when numerical solutions are desired, the system (7) must be truncated to a finite system along with the number of components of $\mathbf{F}^b$, and the formal inversion of $\mathcal{C}^b$ must be reduced to a workable procedure. Before going on to consider such truncations, we can place the system into a standard form occasionally useful for formal theoretical considerations and which also shows the general overall structure of the system (1). Thus we first agree to write:

$$"\mathbf{F}" \text{ for } (\ldots, \mathbf{F}^{-2}, \mathbf{F}^{-1}, \mathbf{F}^{0}, \mathbf{F}^{1}, \mathbf{F}^{2}, \ldots)$$

$$"\mathbf{G}_\eta" \text{ for } (\ldots, \mathbf{G}^{-2}_\eta, \mathbf{G}^{-1}_\eta, \mathbf{G}^{0}_\eta, \mathbf{G}^{1}_\eta, \mathbf{G}^{2}_\eta, \ldots)$$

and finally:

"$\mathcal{B}$" for diag $(\ldots, \mathcal{B}^{-2}, \mathcal{B}^{-1}, \mathcal{B}^{0}, \mathcal{B}^{1}, \mathcal{B}^{2}, \ldots)$

where "diag" denotes a diagonal block matrix with $\mathcal{B}^i$ as the i^{th} block matrix along the diagonal. Then the system (7) takes the generic form:

$$\boxed{-\frac{d\mathbf{F}}{dz} = \mathbf{F}\,\mathcal{B} + \mathbf{G}_\eta} \tag{10}$$

This is the desired vectorial version of the system (1), showing the overall linear form of the system, a form reminiscent of the equation of transfer without the path function term. Thus we see from still another vantage point that the net effect of the spherical harmonic method is the removal of the complex directional dependence of the radiance field generated by the presence of the path function term N_* in the general equation of transfer.

A Truncated Solution Procedure

As an illustration of the use of (7) in practice, we consider the case of an arbitrarily stratified source-free plane-parallel medium. Thus in (7) we set:

$$\mathbf{G}_\eta^b = 0 \tag{11}$$

for every integer b, $|b| \geq 0$. This is a commonly occurring radiometric situation in most natural media in geophysical optics, so that the present illustration retains a wide range of applicability. The effect of condition (11) is rather far-reaching. To see this effect, observe that by the definition of C(a,b) we have:

$$C(a,-b) = C(a,b) \quad .$$

From this it follows that, formally

$$\mathcal{C}^{-b} = \mathcal{C}^{b} \quad \text{and so} \quad \mathcal{B}^{-b} = \mathcal{B}^{b} \quad . \tag{12}$$

Thus we need only consider:

$$-\frac{d\mathbf{F}^b}{dz} = \mathbf{F}^b\,\mathcal{B}^b \quad , \quad b \geq 0 \tag{13}$$

Now the truncation procedure which we intend to apply to (13) may best be described by returning to the original system (1) and keeping in mind the diagram of Fig. 6.2. This return to (1) is also desirable, so as to bypass the formal inversion procedure leading to (13). It is clear from the diagram in Fig. 6.2 that a truncation may take place at the m^{th} row, in the sense that no unknown functions F_a^b will be

allowed in the system which have indices $a > m$. Then the truncated autonomous system of equations associated with $b = 0$ is:

$$\left.\begin{array}{ll}
a = 0: & C(0,0)\,\dfrac{\partial F^0_1}{\partial z} = (-\alpha+\sigma_0)\,F^0_0 \\[2ex]
a = 1: & C(1,0)\,\dfrac{\partial F^0_0}{\partial z} + C(2,0)\,\dfrac{\partial F^0_2}{\partial z} = (-\alpha+\sigma_1)\,F^0_1 \\[2ex]
a = 2: & C(2,0)\,\dfrac{\partial F_1}{\partial z} + C(3,0)\,\dfrac{\partial F^0_3}{\partial z} = (-\alpha+\sigma_2)\,F^0_2 \\[2ex]
\cdots & \\
\cdots & \\
\cdots & \\
a = m-1: & C(m-1,0)\,\dfrac{\partial F^0_{m-2}}{\partial z} + C(m,0)\,\dfrac{\partial F^0_m}{\partial z} = (-\alpha+\sigma_{m-1})\,F^0_{m-1} \\[2ex]
a = m: & C(m,0)\,\dfrac{\partial F^0_{m-1}}{\partial z} = (-\alpha+\sigma_m)\,F^0_m
\end{array}\right\}\quad \textbf{(14)}$$

The effect of the truncation becomes clear on inspection of the equation corresponding to the case $a = m$. The derivative of F^0_{m+1} is omitted from the equation for this case. Thus in the system displayed above there are $(m+1)$ differential equations and $m+1$ unknown functions: F^0_j, $j = 0, 1, \ldots, m$.

The truncated autonomous system of equations associated with $b = 1$ is:

$$\left.\begin{array}{ll}
a = 1: & C(2,1)\,\dfrac{\partial F^1_2}{\partial z} = (-\alpha+\sigma_1)\,F^1_1 \\[2ex]
a = 2: & C(2,1)\,\dfrac{\partial F^1_1}{\partial z} + C(3,1)\,\dfrac{\partial F^1_3}{\partial z} = (-\alpha+\sigma_2)\,F^1_2 \\[2ex]
a = 3: & C(3,1)\,\dfrac{\partial F^1_2}{\partial z} + C(4,1)\,\dfrac{\partial F^1_4}{\partial z} = (-\alpha+\sigma_3)\,F^1_3 \\[2ex]
\cdots & \\
\cdots & \\
\cdots & \\
a = m-1: & C(m-1,1)\,\dfrac{\partial F^1_{m-2}}{\partial z} + C(m,1)\,\dfrac{\partial F^1_m}{\partial z} = (-\alpha+\sigma_{m-1})\,F^1_{m-1} \\[2ex]
a = m: & C(m,1)\,\dfrac{\partial F^1_{m-1}}{\partial z} = (-\alpha+\sigma_m)\,F^1_m
\end{array}\right\}\quad (15)$$

Here the system associated with $b = 1$ consists of m differential equations in m unknown functions: F^1_j, $j = 1, \ldots, m$. In general the system associated with b, where $b \leq m$, consists of $m+1-b$ differential equations in the $m+1-b$ unknown functions F_j, $j = b, b+1, \ldots, m$. Thus for the case $b = m-1$,

we have two equations:

$$\left.\begin{array}{lll} a = m-1: & C(m,m-1)\,\dfrac{\partial F_m^{m-1}}{-\partial z} = (-\alpha+\sigma_{m-1})\,F_{m-1}^{m-1} \\ a = m: & C(m,m-1)\,\dfrac{\partial F_{m-1}^{m-1}}{-\partial z} = (-\alpha+\sigma_m)\,F_m^{m-1} \end{array}\right\} \quad (16)$$

Finally, for the case b = m, there is only one equation, namely: a

$$a = m \qquad 0 = (-\alpha+\sigma_m)\,F_m^m \quad (17)$$

whence $F_m^m = 0$, provided $(-\alpha+\sigma_m) \neq 0$.

Once the $m^2+2m+1 = (m+1)^2$ functions F_a^b have been obtained, where $0 \leq a \leq m$, and $|b| \leq a$, the associated representation of $N(x,\xi)$ is, according to the general pattern (24) of Sec. 6.3:

$$\boxed{N(x,\xi) = \sum_{a=0}^{m}\ \sum_{b=-a}^{a} F_a^b(x)\ \Phi_a^b(\xi)} \quad (18)$$

Equation (18) is the requisite *m^{th} order* spherical harmonic approximation to the radiance function N on a stratified plane-parallel source-free optical medium in the steady state.

Vector Form of the Truncated Solution

It is of interest to place the truncated system (14) to (17) into the compact form of (13). Thus let us write:

$$\text{"}\mathbf{F}(b,m)\text{"} \quad \text{for} \quad (F_b^b,\ F_{b+1}^b,\ \ldots,\ F_m^b) \quad (19)$$

$\mathbf{F}(b,m)$ is a function which assigns to each depth z in the plane-parallel medium the (m+1)-component vector $\mathbf{F}(b,m;a)$, i.e.,

$$F_b^b(z),\ F_{b+1}^b(z),\ \ldots,\ F_m^b(z) \quad .$$

By studying the general forms of (14) to (17), we see that the truncated associate of $\mathcal{C}^b$ in (5) is the $(m-b+1)\times(m-b+1)$ matrix:

$$\begin{bmatrix} 0 & C(b+1,b) & 0 & \cdots & 0 & 0 \\ C(b+1,b) & 0 & C(b+2,b) & \cdots & 0 & 0 \\ 0 & C(b+2,b) & 0 & \cdots & 0 & 0 \\ 0 & 0 & C(b+3,b) & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C(m-1,b) & 0 \\ 0 & 0 & 0 & \cdots & 0 & C(m,b) \\ 0 & 0 & 0 & \cdots & C(m,b) & 0 \end{bmatrix} \tag{20}$$

which we shall denote by "$\mathcal{C}(b,m)$". This matrix is invertible whenever (m-b) is odd as we shall see below. Furthermore, if we write:

$$\text{"}\mathcal{A}(m)\text{" for } \begin{bmatrix} -\alpha+\sigma_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha+\sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\alpha+\sigma_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\alpha+\sigma_{m-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\alpha+\sigma_m \end{bmatrix} \tag{21}$$

then the general representative of the systems of equations (14) to (17) is of the form:

$$-\frac{d}{dz}\, \mathbf{F}(b,m)\, \mathcal{C}(b,m) = \mathbf{F}(b,m)\, \mathcal{A}(m) \tag{22}$$

Finally, if we write:

$$\text{"}\mathcal{B}(b,m)\text{" for } \mathcal{A}(m)\, \mathcal{C}^{-1}(b,m) \tag{23}$$

we have:

$$\boxed{\begin{gathered} -\frac{d}{dz}\, \mathbf{F}(b,m) = \mathbf{F}(b,m)\, \mathcal{B}(b,m) \\ 0 \le b \le m;\ m-b \text{ odd} \end{gathered}} \tag{24}$$

which is the desired vector form of the system (14) to (17) of m^{th} order spherical harmonic equations. We have now reached the stage where the system (1) is in a form amenable

to solution by any of several well-known theoretical or numerical techniques in the theory of ordinary differential equations (see, e.g., [23] or [47]). Of course (1) itself can always be programmed directly for solution.

There is one instance of (24) whose solution can be written down immediately in "closed form," namely the case where α and σ are independent of depth; in other words, for the case of an homogeneous medium X. Then, if we write:

$$\exp\left\{\mathcal{B}(b,m)\right\} \quad \text{for} \quad \sum_{j=0}^{\infty} \frac{\mathcal{B}^j(b,m)\; z^j}{j!} \tag{25}$$

where $\mathcal{B}^j(b,m)$ is the j^{th} power of the matrix $\mathcal{B}(b,m)$, and denote the value of $\mathbf{F}(b,m)$ at z by "$\mathbf{F}(b,m;z)$" then:

$$\boxed{\begin{array}{c} \mathbf{F}(b,m;z) = \mathbf{F}(b,m;0)\ \exp\left\{-\mathcal{B}(b,m)\ z\right\} \\ 0 \le b \le m;\ m-b = 1,\ 3,\ 5,\ \ldots \end{array}} \tag{26}$$

Using the theories of [37], (26) may on the one hand be put into closed algebraic terms using the Jordan canonical forms of matrices; and on the other, (26) may be programmed for direct evaluation on general-purpose electronic computers using the techniques, for example, in [23].

To facilitate computations of $\mathbf{F}(b,m)$ using (26), we may arrange matters so that the inverse of $\mathcal{C}(b,m)$ can be written down by inspection whenever it exists. This may be done as follows. First we verify the fact already noted, that $\mathcal{C}(b,m)$ has an inverse whenever m-b is an odd integer. For example, when m-b = 1, and b $\ge$ 0

$$\mathcal{C}(b,m) = \begin{bmatrix} 0 & C(b+1,b) \\ C(b+1,b) & 0 \end{bmatrix}$$

then

$$\det \mathcal{C}(b,m) = -\ C^2(b+1,b) = \frac{-1}{(2b+3)} \neq 0$$

where "det A" denotes the determinant of a matrix A. Hence $\mathcal{C}(b,m)$ has an inverse. Again, when m-b = 2, and b $\ge$ 0

$$\mathcal{C}(b,m) = \begin{bmatrix} 0 & C(b+1,b) & 0 \\ C(b+1,b) & 0 & C(b+2,0) \\ 0 & C(b+2,b) & 0 \end{bmatrix}$$

then

$$\det \mathcal{C}(b,m) = 0$$

so that $\mathcal{C}(b,m)$ has no inverse in this case. Once more, for $m - b = 3$, $b \geq 0$,

$$\mathcal{C}(b,m) = \left[\begin{array}{cc|cc} 0 & C(b+1,b) & 0 & 0 \\ C(b+1,b) & & C(b+2,b) & 0 \\ \hline 0 & C(b+2,0) & 0 & C(b+3,b) \\ 0 & 0 & C(b+3,) & 0 \end{array}\right]$$

and

$$\det \mathcal{C}(b,m) = C^2(b+1,b)\ C^2(b+3,b)$$

$$= \frac{3}{(2b+5)(2b+7)} \neq 0$$

The pattern forming should now be clear. By induction we have, for integers $b \geq 0$, $p \geq 0$ such that $m - b = 2p + 1$.

$$\det \mathcal{C}(b,m) = (-1)^{p+1} \prod_{j=0}^{p} C^2(b+(2j+1),b) \neq 0 \qquad (27)$$

We next introduce the permutation matrix P which permutes the m-b+1 rows of $\mathcal{C}(b,m)$, where $m - b = 2p + 1$, $p \geq 0$, in such a way as to *near-diagonalize* $\mathcal{C}(b,m)$, in the following sense. Return to $\mathcal{C}(b,m)$ above where $m - b = 1$ and note that we can diagonalize it by interchanging its two rows. Similarly, by interchanging the rows of $\mathcal{C}(b,m)$ where $m - b = 3$ in pairs, starting with the first two rows, then going on to interchange the second pair of rows (i.e., row three and four) we obtain:

$$P\mathcal{C}(b,m) = \left[\begin{array}{cc|cc} C(b+1,b) & 0 & C(b+2,b) & 0 \\ 0 & C(b+1,b) & 0 & 0 \\ \hline 0 & 0 & C(b+3,b) & 0 \\ 0 & C(b+2,0) & 0 & C(b+3,b) \end{array}\right]$$

where

$$P = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The general structure of P for the case of an arbitrary $m - b$ ($= 2p + 1$) should now be evident: P is a $2(p+1) \times 2(p+1)$ matrix obtained from the identity matrix I of the same order by interchanging the rows of I in pairs, as illustrated by the special case just considered. The utility of the permutation P rests in the fact that the inverse of $P\mathcal{C}(b,m)$ where $m - b = 2p + 1$, is readily written down by inspection. To see how the inversion proceeds, consider once again the case of $m - b = 3 = 2p + 1$ (so that $p = 1$). To simplify the illustration, we shall write "C_j" for $C(b+j,b)$, with "b" understood. Inspection of $P\mathcal{C}(b,m)$, with $m-b = 3$, shows that its inverse must have the same overall structure as $P\mathcal{C}(b,m)$ itself and whose main diagonal consists simply of elements of the form $1/C_j$. With this in mind, we may write:

$$[P\mathcal{C}(b,m)][P\mathcal{C}(b,m)]^{-1} = \left[\begin{array}{cc|cc} C_1 & 0 & C_2 & 0 \\ 0 & C_1 & 0 & 0 \\ \hline 0 & 0 & C_3 & 0 \\ 0 & C_2 & 0 & C_3 \end{array}\right]\left[\begin{array}{cc|cc} \frac{1}{C_1} & 0 & x_1 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{C_3} & 0 \\ 0 & x_2 & 0 & \frac{1}{C_3} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = I$$

As yet the entires x_1, x_2 of the matrix are not known. However, it is clear that x_1, x_2 satisfy the conditions:

$$x_1 C_1 + \frac{C_2}{C_3} = 0$$

$$x_2 C_3 + \frac{C_2}{C_1} = 0$$

whence

$$x_1 = -\frac{C_2}{C_1 C_3}$$

$$x_2 = -\frac{C_2}{C_1 C_3}$$

As another example, let $m - b = 5 = 2p + 1$ (so that $p = 2$). Once again the overall structure of $[P\mathcal{C}(b,m)]^{-1}$ is the same

as $P\mathcal{C}(b,m)$; i.e., *near diagonal*, where P is now the requisite 6 x6 row permutation matrix. To find $[P\mathcal{C}(b,m)]^{-1}$, we write:

$[P\mathcal{C}(b,m)]\ [P\mathcal{C}(b,m)]^{-1} =$

$$\left[\begin{array}{cc|cc|cc} C_1 & 0 & C_2 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & C_3 & 0 & C_4 & 0 \\ 0 & C_2 & 0 & C_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & C_5 & 0 \\ 0 & 0 & 0 & C_4 & 0 & C_5 \end{array}\right] \left[\begin{array}{cc|cc|cc} \frac{1}{C_1} & 0 & x_1 & 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{C_3} & 0 & x_2 & 0 \\ 0 & x_3 & 0 & \frac{1}{C_3} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{C_5} & 0 \\ 0 & 0 & 0 & x_4 & 0 & \frac{1}{C_5} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = I$$

The remaining entries $x_1, \ldots, x_4$ are now readily determined as in the case of $p = 1$. By direct inspection:

$$x_1 = x_3 = -\frac{C_2}{C_1 C_3}$$

$$x_2 = x_4 = -\frac{C_4}{C_3 C_5}$$

These two examples for the cases $p = 1,2$ clearly indicate the nature of $[P\mathcal{C}(b,m)]^{-1}$ with $m-b = 2p+1$ for general integers $p \geq 0$. The general rule may be phrased as follows: the main diagonal of $[P\mathcal{C}(b,m)]^{-1}$ consists of elements of the form $1/C(b+(2j+1),\ b)$ arranged successively in pairs for $j = 0, 1, \ldots, p$. The nonzero off-diagonal elements in $[P\mathcal{C}(b,m)]^{-1}$ occur in exactly the same places as in $P\mathcal{C}(b,m)$, and each may be obtained by dividing the corresponding entry C_j of $P\mathcal{C}(b,m)$

by (-1) times $C_k C_\ell$, where C_k and C_ℓ are, respectively, the elements of $P\mathcal{C}(b,m)$ in the same row and column as C_i. The reader should now construct the $[P\mathcal{C}(b,m)]^{-1}$ for p = 3 to test this rule. What is the rule's general form?

Finally, we can rearrange (26) so as to specifically include within the formalism the preceding simple inversion procedure. Returning to (22), we can write:

$$-\frac{d}{dz}\left[\left(\mathbf{F}(b,m)P^{-1}\right) P\mathcal{C}(b,m)\right] = \left(\mathbf{F}(b,m)P^{-1}\right)\left(P\mathcal{Q}(m)\right)$$

Writing

$$\text{"}\mathbf{G}(b,m)\text{"} \quad \text{for} \quad \mathbf{F}(b,m)P^{-1} \tag{28}$$

$$\text{"}\mathcal{D}(b,m)\text{"} \quad \text{for} \quad [P\mathcal{Q}(m)]\;[P\mathcal{C}(b,m)]^{-1} \tag{29}$$

we have

$$\boxed{\begin{array}{c} -\frac{d}{dz}\,\mathbf{G}(b,m) = \mathbf{G}(b,m)\,\mathcal{D}(b,m) \\ 0 \leq b \leq m;\; m-b = 1, 3, 5, \ldots \end{array}} \tag{30}$$

as the present counterpart to (24). The inverse $[P\mathcal{C}(b,m)]^{-1}$ is the one whose simple rule of formation we have generated in the preceding discussion. Then, corresponding to (26), we have:

$$\boxed{\begin{array}{c} \mathbf{G}(b,m;z) = \mathbf{G}(b,m;0)\exp\{-\mathcal{D}(b,m)z\} \\ 0 \leq b \leq m;\; m-b = 1, 3, 5, \ldots \end{array}} \tag{31}$$

Because of the autonomy of these equations with respect to b, we can vary the truncation parameter m for each given b, so as always to have m-b odd, and therefore, to always have the algorithm (31) at hand. Suppose, for example, we wish to find all F_a^b with $a \leq 4$, as indicated by the diagram in Fig. 6.2, and so as to have the representation of $N(x,\xi)$ in (18) for the case m = 4. Thus we are to find $(4+1)^2 = 25$ functions in all. In solving for the family $\{F_a^o\}$ we accordingly may truncate at F_5^o (rather than F_4^o) and solve for F_a^o, a = 0, 1, 2, 3, 4, 5 using (31), taking advantage of the oddness of m-b = 5-0 = 5. In solving for the family $\{F_a^1\}$, we use (31) directly since now m-b = 4-1 = 3. A similar tactic is employed for extending by one additional member the family $\{F_a^2\}$, as in the case of F_a^o, and so on, to the end of the computation procedure.

Equations (26) and (31) are the final forms of the m^{th} order spherical harmonic equations we shall study in this work. Having deduced (26), (or its variant (31)) we reach

the threshold of the invariant imbedding domain of radiative transfer theory. Thus the equation (26), say, may be viewed on the one hand, as the logical culmination of the train of deductions begun in Sec. 6.2 in the development of the classical spherical harmonic method; and on the other hand (26) forms a bridge between the classical method of solution of the equation of transfer and the invariant imbedding techniques for the solution of the equation of transfer. These latter techniques will be considered in Sec. 7.10.

Summary

In the preceding four Secs. 6.1 to 6.4 the spherical harmonic method is developed and applied after an appropriate motivation of the method in Sec. 6.1. The main purpose of the discussions is to make clear the fundamental ideas on which the method rests, in particular the general role of the orthonormal family of functions used to represent the radiance function as a sum of products of purely spatial and directional terms. This was done in Secs. 6.2 and 6.3. To show the applicability of the method to the case of plane-parallel media, the setting of greatest utility in the study of hydrologic and meteorologic optics, the discussion of the present section is added to the general remarks. In particular, equation sets (14) to (17) above explicitly exhibit the truncated forms of the spherical harmonic equations, where the truncation arbitrarily sets to zero all functions F_a^b with indices $a > m$. The resultant system (24) can be used to solve for the unknown complex valued functions F_a^b, $0 \leq a, \leq m$, $|b| \leq a$. To solve (24) directly we must know $\bar{N}^o$ (in (31) or (32) of Sec. 6.3) from experiments. If N^o is to be found theoretically, we may use invariant imbedding methods which will give the aerosol's or hydrosol's reflectance to incident light (Volume IV, *et seq.*).

6.5 Three Approaches to Diffusion Theory

The term "diffusion theory" in the context of radiative transfer theory denotes a discipline based on not any single equation, but rather a collection of more or less loosely interconnected theories each springing from some analytic expression which, in turn, is based on the fundamental equation of transfer. For our present purposes we may broadly classify this collection of diffusion theories into two main groups: the *approximate* and the *exact* theories. A diffusion theory is approximate to a greater or lesser degree depending on the amount of modification undergone by the analytic structure of the equation of transfer as the equation is subject to simplifying assumptions. In the present section our purpose is to approach this complex of diffusion theories from three different directions so as to gain a useful overall perspective of the sub-discipline of diffusion theory within general radiative transfer theory. In particular we shall approach one of the more useful approximate diffusion theories (called *classical diffusion* theory, for reasons which will eventually become clear) by starting from the equation of transfer and

proceeding to transform the equation by adopting the assumption of Fick's law for diffusing photons. Then we shall start again, this time proceeding via spherical harmonic theory which, depending on the order of terms retained in the basic system (27) of Sec. 6.3, opens up a multitude of paths into the domain of approximate diffusion theory. This approach serves to show the extremely large number of diffusion-type theories generally possible, and to throw light on the classical diffusion theory by appropriately placing the latter in the hierarchy of approximate diffusion theories springing from the system of spherical harmonic equations of Sec. 6.3. Finally, we start afresh once more from the equation of transfer and develop the basic equation for an important exact diffusion theory which applies rigorously to optical media whose volume scattering functions σ are independent of the directions ξ' and ξ.

The Approach via Fick's Law

We begin with the general time-dependent equation of transfer (re (4) of Sec. 3.15) with source term in a generally inhomogeneous optical medium X:

$$\frac{1}{v}\frac{\partial N(x,\xi,t)}{\partial t} + \xi \cdot \nabla N(x,\xi,t) = -\alpha(x,t)\,N(x,\xi,t) + N_*(x,\xi,t) + N_\eta(x,\xi,t) \qquad (1)$$

Diffusion theory is characteristically interested in the description of the scalar irradiance $h(x,t)$ rather than the radiance $N(x,\xi,t)$. That is, the density of the total flow at x in all directions is of interest rather than the density of the flow in each direction ξ at x. Thus we are led to integrate each term of (1) over direction space Ξ. The reduction of the resulting integrated form of (1) is facilitated by recalling from (4) of Sec. 4.2 that:

$$\alpha(x,t) = a(x,t) + s(x,t) \qquad (2)$$

and from (2) of Sec. 2.8 that we write:

$$\text{"}\mathbf{H}(x,t)\text{"} \quad \text{for} \quad \int_\Xi N(x,\xi,t)\,\xi\, d\Omega(\xi) \quad , \qquad (3)$$

where $\mathbf{H}(x,t)$ is the *vector irradiance* at x at time t.

The reduced integrated form of (1) is:

$$\frac{1}{v}\frac{\partial h(x,t)}{\partial t} + \nabla \cdot \mathbf{H}(x,t) = -\,a(x,t)\,h(x,t) + h_\eta(x,t) \qquad (4)$$

where we have written:

$$"h_\eta(x,t)" \quad \text{for} \quad \int_\Xi N_\eta(x,\xi,t)\, d\Omega(\xi) \quad .$$

Equation (4) lacks utility in our present efforts to describe the scalar irradiance throughout X. The presence of the divergence term for the vector irradiance blocks immediate usage of (4) in this respect: If, somehow, $\nabla \cdot \mathbf{H}$ could be replaced by a single function of h, then the resulting form of (4) would be a useful statement involving only scalar irradiance. It is at this point that the customary appeal to *Fick's law* of diffusion is made. This law states that, for some nonnegative valued function D, on X:

$$\boxed{\mathbf{H}(x,t) = - D(x,t)\ \nabla h(x,t)} \tag{5}$$

for every t in some time interval. In other words, at each point x and time t, the vector $\mathbf{H}(x,t)$ has the direction of the negative of the gradient of the scalar irradiance field h. In still other terms, $\mathbf{H}$ has the direction from the greatest to the smallest values of h in the neighborhood of a point. The spatial and temporal variation of D is required to be quite mild, and for essentially all practical applications D is assumed constant. The types of media for which Fick's law is a reasonably good description of the state of affairs between $\mathbf{H}$ and h are those for which the scattering attenuation ratio ρ is large, say on the order of 0.6 and above. All other things being equal the closer ρ is to 1 (i.e., the larger the proportion of scattering compared to absorption), the closer does Fick's law describe $\mathbf{H}$ in terms of h. Furthermore, Fick's law, all other things being equal, increases in accuracy with distance from the boundaries and highly directional or concentrated sources of the medium until the effects of these boundaries and sources have disappeared. Any physical breakdown of a formula of the resultant theory is eventually traceable to a marked inapplicability of Fick's law. Using (5) in (4), we have:

$$\boxed{\frac{1}{v} \cdot \frac{\partial h(x,t)}{\partial t} - \nabla \cdot (D(x,t)\ \nabla h(x,t)) = - a(x,t)\ h(x,t) + h_\eta(x,t)} \tag{6}$$

Equation (6) is the desired *scalar diffusion equation* for scalar irradiance h. D is the *diffusion function* (or *constant,* as the case may be), a is the volume absorption function, and h_η the *emission* or *source* term for the equation. The diffusion theory based on (6) is the classical (*scalar*) *diffusion theory.* When D is assumed constant over the space

X and a given time interval, an assumption which henceforth shall be in force, (6) may be written:

$$\boxed{\frac{1}{v}\frac{\partial h}{\partial t} - D\,\nabla^2 h = -\,ah + h_\eta} \qquad (7)$$

Equation (7) has the Gestalt of the diffusion equation of classical heat conduction and other diffusion phenomena with source term (h_η) and annihilation term ($-ah$), hence the mathematics of the diffusion of photons as governed by (7) is identical to that of the diffusion of heat and other classical diffusion phenomena, the theory of which is thoroughly understood. Therefore (7) may possibly be applied to such problems as describing the transient light field set up by pulsed sources. Equation (7) and related equations are studied further in Table 1 below, and in Sec. 6.6.

The Approach via Spherical Harmonics

The next approach to diffusion theory we shall describe is that via the spherical harmonic theory developed in Sec. 6.4. It will be seen that the approach can take place on several levels of generality and in an infinite number of directions on each level. We shall begin our discussion with one of the simpler directions of approach on a very practical level, the goal being once again the classical scalar diffusion equation (7). However, now awaiting us at the goal is the added bonus of a theoretical representation for the diffusion constant D and a formula describing the radiance function in a general diffusing medium in terms of the vector and scalar irradiances.

In our present approach to diffusion theory we shall be guided by the following two special principles concerning the components F_a^b of the spherical harmonic representation of the radiance function:

(i) All components F_a^b other than F_0^0, F_1^{-1}, F_1^1 are set equal to zero in the system (27) of Sec. 6.3. All components of $F_{\eta,a}^b$ other than $F_{\eta,0}^b$ are zero.

(ii) All time derivatives of the components F_a^b other than F_0^0 are set equal to zero in the system (27) of Sec. 6.3.

The reason for these two special principles stems ultimately from our intuitive conception of a diffusive flow of material (or light) particles: (i) the amount of diffusive flow about a point varies mildly from direction to direction, and (ii) the overall directional structure of the flow itself varies mildly from moment to moment. With this intuitive conception in mind, the rules of action stated in (i) and (ii) above are arrived at by pairing F_0^0 with h and by identifying the components F_1^{-1}, F_1^0, F_1^1 as the first three of an infinite set of components describing the overall directional flow of radiant energy at a point. The basis of this pairing of F_0^0

with h is as follows. By (6) and (25) of Sec. 6.3 we have the definitional identity:

$$F_0^0(x,t) = \int_\Xi N(x,\xi,t)\ \overline{\Phi_0^0(\xi)}\ d\Omega(\xi)$$

$$= A_0^0 \int_\Xi N(x,\xi,t)\ P_0^0(\xi)\ d\Omega(\xi)$$

$$= A_0^0\ h(x,t) = (4\pi)^{-1/2}\ h(x,t) \quad . \qquad (8)$$

The fact that the three components F_1^{-1}, F_1^0, F_1^1 are associated with the overall directional structures of the radiant flux is established by first noting that:

$$\mathbf{H}(x,t) = \int_\Xi N(x,\xi,t)\ \xi d\Omega(\xi)$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_n^m(x,t) \int_\Xi \Phi_n^m(\xi)\ \xi d\Omega(\xi) \qquad (9)$$

Furthermore, we have (cf. Fig. 2.4):

$$\xi = \sin\theta \cos\phi\,\mathbf{i} + \sin\theta\ \sin\phi\ \mathbf{j} + \cos\theta\ \mathbf{k} \qquad (10)$$

If we could now express the quantities sin θ cos θ, sin θ sin ϕ and cos ϕ as linear combinations of the Φ_n^m, then we could directly evaluate the integral in (9) using the orthonormality properties of the Φ_n^m. Toward this end we recall that $\sin\theta = (1-\cos^2\theta)^{1/2} = (1-\mu^2)^{1/2}$. Furthermore, an examination of any list of associated Legendre functions reveals that:

$$L_1^1(\mu) = -\,2P_1^{-1}(\mu) = (1-\mu^2)^{1/2} \quad .$$

Then:

$$\sin\theta\ (\cos\phi + i\ \sin\phi) = P_1^1(\mu)\ e^{i\phi}$$

$$= (A_1^1\ P_1^1(\mu)\ e^{i\phi})/A_1^1$$

$$= \Phi_1^1(\xi)/A_1^1$$

Similarly:

$$\sin\theta\,(\cos\phi - i\sin\phi) = -2P_1^{-1}(\mu)\,e^{-i\phi}$$

$$= (-2\,A_1^{-1}\,P_1^{-1}(\mu)\,e^{-i\phi})/A_1^{-1}$$

$$= -2\,\Phi_1^{-1}(\xi)/A_1^{-1}$$

$$= -\,\Phi_1^{-1}(\xi)/A_1^{1}$$

From these expressions we deduce that:

$$\sin\theta\cos\phi = \frac{1}{2A_1^1}\left(\Phi_1^1(\xi) - \Phi_1^{-1}(\xi)\right) \tag{11}$$

$$\sin\theta\sin\phi = \frac{1}{2iA_1^1}\left(\Phi_1^1(\xi) + \Phi_1^{-1}(\xi)\right) \tag{12}$$

Finally, we observe that:

$$\cos\theta = \mu = P_1(\mu) = P_1^0(\mu)$$

$$= A_1^0\,P_1^0(\mu)\,e^{io\phi}A_1^0 = \Phi_1^0(\xi)/A_1^0 \tag{13}$$

Using (11) to (13) in (10), we have the requisite representation of ξ as a linear combination involving only members Φ_n^m of the orthonormal family. The conjugates of Φ_n^m are obtained using (8) of Sec. 6.3. As a result, (9) reduces immediately to:

$$\boxed{\begin{aligned} \mathbf{H}(x,t) = {} & \frac{1}{2A_1^1}\left[F_1^1(x,t) - F_1^{-1}(x,t)\right]\mathbf{i} \\ & - \frac{1}{2iA_1^1}\left[F_1^1(x,t) + F_1^{-1}(x,t)\right]\mathbf{j} \\ & + \frac{1}{A_1^0}\,F_1^0(x,t)\,\mathbf{k} \end{aligned}} \tag{14}$$

This is the desired representation of the vector irradi- $\mathbf{H}(x,t)$ in terms of the spherical harmonic components F_a^b of the radiance function N. The representation reveals the rôle played by the three components F_1^{-1}, F_1^0, F_1^1 in the description of the overall directional structure of the light field (see also (29) below).

With the basis for the two special principles (i) and (ii) now reasonably well established, we next apply these special principles to the system (27) of Sec. 6.3. According to principle (i), we need consider only the cases a = 0, 1. According to principle (ii), all time derivatives, except that of F_0^0, vanish. The resultant set of four equations is:

$$\frac{1}{v}\frac{\partial F_0^0}{\partial t} + C(1,0)\frac{\partial F_1^0}{\partial x_3} - \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)B(1,1)F_1^{-1} + \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)B(1,1)\,F_1^1$$

$$= (-\alpha + \sigma_0)\,F_0^0 + F_{\eta,0}^0 \qquad (a = 0,\ b = 0 \text{ in } F_a^b) \tag{15}$$

$$-\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)B(1,1)\,F_0^0 = (-\alpha + \sigma_1)\,F_1^{-1}$$

$$(a = 1,\ b = -1 \text{ in } F_a^b) \tag{16}$$

$$C(1,0)\,\frac{\partial F_0^0}{\partial x_3} = (-\alpha + \sigma_1)\,F_1^0$$

$$(a = 1,\ b = 0 \text{ in } F_a^b) \tag{17}$$

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)B(1,1)\,F_0^0 = (-\alpha + \sigma_1)\,F_1^1$$

$$(a = 1,\ b = 1 \text{ in } F_a^b) \tag{18}$$

Our present goal is to obtain a single diffusion equation for h(x,t) from the system (15) to (18). In view of the connection between F_0^0 and h stated in (8), we see that the goal will be in sight if we use (16) to (18) to replace each occurrence of F_1^{-1}, F_1^0, F_1^1 in (15) in terms of F_0^0. Thus the term:

$$C(1,0)\,\frac{\partial F_1^0}{\partial x_3}$$

in (15), with the help of (17), becomes:

$$\frac{C^2(1,0)}{(-\alpha+\sigma_1)}\,\frac{\partial^2 F_0^0}{\partial x_3^2} = \frac{1}{3(-\alpha+\sigma_1)}\,\frac{\partial^2 F_0^0}{\partial x_3^2} \tag{19}$$

Further the term:

$$-\frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right) \quad B(1,1)\ F_1^{-1}$$

in (15), with the help of (16), becomes:

$$\frac{1}{4}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)\frac{B^2(1,1)}{(-\alpha+\sigma_1)}F_0^0 = \frac{1}{6}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\frac{1}{(-\alpha+\sigma_1)} \quad F_0^0$$

In a similar way the term:

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) \quad B(1,1)\ F_1^1$$

in (15), with the help of (18), becomes:

$$\frac{1}{6}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\frac{1}{(-\alpha+\sigma_1)} \quad F_0^0$$

Combining these terms in (15), the result is:

$$\frac{1}{v}\,\frac{\partial F_0^0}{\partial t} + \frac{1}{3(-\alpha+\sigma_1)}\left[\frac{\partial^2 F_0^0}{\partial x_1^2} + \frac{\partial^2 F_0^0}{\partial x_2^2} + \frac{\partial F_0^0}{\partial x_3^2}\right] = (-\alpha+\sigma_0)\ F_0^0 + F_{\eta,0}^0 \quad . \tag{20}$$

We are now ready to pair off the terms in (20) with their correspondents in (7). Multiplying each side of (20) by $(4\pi)^{1/2}$ and using (8), we can replace each occurrence of "F_0^0" in (20) by "h". Next, by (15) of Sec. 6.3, we have:

$$\sigma_0(x;t) = 2\pi\int_{-1}^{1} \sigma(x;\mu;t)\ P_0(\mu)\ d\mu$$

$$= \int_{\Xi} \sigma(x;\xi';\xi;t)\ d\Omega(\xi)$$

$$= s(x,t) \quad .$$

In other words, σ_0 in (20) is the volume total scattering coefficient. Hence:

$$-\alpha+\sigma_0 = -\ a$$

by virtue of (2). Finally, from (29) of Sec. 6.3 and the definition of h_η in (4), we have:

$$F^0_{\eta,0} = h_\eta \quad .$$

In view of these observations, we may say that the structure of equation (20) is identical with that of (7). Therefore the diffusion coefficient D in (7) is represented by the relation:

$$\boxed{D = \frac{1}{3(\alpha-\sigma_1)}} \tag{21}$$

where α is the volume attenuation coefficient and σ_1 is defined as in (15) of Sec. 6.3 (setting j = 1). This representation of D rests on the basis of the spherical harmonic decomposition of the equation of transfer *subject to the special principles (i) and (ii) stated above which fix the level of approximation of the spherical harmonic decomposition.* In sum, then, the left side of (21) arises when we approach diffusion theory via Fick's law; the right side arises when we approach diffusion theory via the spherical harmonic method. At the point where the twain shall meet, we generate (21).

There are several alternate but equivalent forms of (21) arising in practice. For example, if we write

$$\text{"}\bar{\mu}(x,t)\text{"} \quad \text{for} \quad \frac{2\pi}{s(x,t)} \int_{-1}^{1} \sigma(x;\mu;t)\ \mu d\mu \tag{22}$$

Then, by (15) of Sec. 6.3, we have:

$$\sigma_1(x;t) = \bar{\mu}(x,t)\ s(x,t) \tag{23}$$

Thus we see that $\bar{\mu}(x,t)$ is a mean value of the cosine $\mu = \cos\theta = \xi \cdot \xi'$ of the scattering angle θ. Another way of writing (22) to see this more clearly is to note that, when isotropy holds:

$$2\pi \int_{-1}^{1} \sigma(x;\mu;t)\ \mu d\mu = \int_{\Xi} \sigma(x;\xi';\xi;t)\ \xi' \cdot \xi\ d\Omega(\xi) \tag{24}$$

Hence (22) becomes:

$$\bar{\mu}(x,t) = \frac{\int_{\Xi} \sigma(x;\xi';\xi;t)\ \xi'\cdot\xi\, d\Omega(\xi)}{\int_{\Xi} \sigma(x;\xi';\xi;t)\ d\Omega(\xi)} \tag{25}$$

and from this the mean value property of $\bar{\mu}(x,t)$ is quite clear; and by a mean value theorem of integral calculus,

$$-1 \leq \bar{\mu}(x,t) \leq 1 \tag{26}$$

For optical media with large forward scattering values for σ, the values of $\bar{\mu}$ are near 1. For media with uniform scattering, i.e., σ independent of ξ' and ξ, the value of $\bar{\mu}$ is 0. For media with predominant backward scattering values, $\bar{\mu}$ has negative values. Thus, in this sense, $\bar{\mu}$ is a measure of the relative amount of the forward or backward scattering occurring in a beam of flux within the medium. Returning now to (21) we use (23) to obtain:

$$\boxed{\begin{aligned} D &= \frac{1}{3(\alpha-\bar{\mu}s)} \\ &= \frac{1}{3\alpha(1-\bar{\mu}\rho)} \\ &= \frac{L_\alpha}{3(1-\bar{\mu}\rho)} \end{aligned}} \tag{27}$$

where ρ is the scattering-attenuation ratio and where "L_α" denotes the *attenuation length* for the medium; that is, we have written "L_α" for $1/\alpha$. Hence the diffusion coefficient has the dimensions of length and in particular is equal to the attenuation length of the medium divided by the factor $3(1-\bar{\mu}\rho)$.

Radiance Distribution in Diffusion Theory

We conclude the discussion of the present approach by deriving the characteristic form of the radiance distribution $N(x,\cdot,t)$ at a point x about which exists a diffusion process with the properties (i) and (ii). Thus, the radiance $N(x,\xi,t)$ at x at time t in the direction ξ is of the general form:

$$\begin{aligned} N(x,\xi,t) = {} & F_0^0(x,t)\ \Phi_0^0(\xi) + F_1^{-1}(x,t)\ \Phi_1^{-1}(\xi) \\ & + F_1^0(x,t)\ \Phi_1^0(\xi) + F_1^1(x,t)\ \Phi_1^1(\xi) \end{aligned} \tag{28}$$

This form follows by using the present diffusion properties (i) and (ii) in (24) of Sec. 6.3. By evaluating each of the eight factors in the four terms of (28), and simplifying, we obtain:

$$\boxed{N(x,\xi,t) = \frac{1}{4\pi}\,[h(x,t) + 3\xi\cdot\mathbf{H}(x,t)]} \qquad (29)$$

Equation (29) displays the relatively mild structure of the radiance distribution associated with a classical diffusion process in an arbitrary optical medium. The greatest radiance occurs in the direction of $\mathbf{H}(x,t)$. In directions ξ perpendicular to $\mathbf{H}(x,t)$ the radiance is simply $h(x,t)/4\pi$. Observe that the overall graphical structure of $N(x,\cdot,t)$ at a point is simply that of a cardioid of revolution with axis along the direction of $\mathbf{H}(x,t)$. Using (5) we may cast (29) into radiometric terms involving $h(x)$ only:

$$\boxed{N(x,\xi,t) = \frac{1}{4\pi}\,[h(x,t) - 3D\,\xi\cdot\nabla h(x,t)]} \qquad (30)$$

As a representative indication of the details of the derivation of (29) from (28), observe that by (8):

$$F_0^0(x,t) = (4\pi)^{-1/2}\,h(x,t)$$

and that:

$$\Phi_0^0(\xi) = A_0^0 P_0^0(\mu)\,e^{io\phi} = (4\pi)^{-1/2}$$

Hence:

$$F_0^0(x,t)\;\Phi_0^0(\xi) = h(x,t)/4\pi \quad . \qquad (31)$$

Furthermore, by (16):

$$F^{-1}(x,t) = -\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)\cdot\left(\frac{2}{3}\right)^{1/2}\cdot F_0^0(x)\cdot\frac{1}{(-\alpha+\sigma_1)}$$

$$= \frac{1}{2}\left(\frac{6}{4\pi}\right)^{1/2} D\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) h(x,t)$$

Also: $$\Phi_1^{-1}(\xi) = A_1^{-1}\,P_1^{-1}(\mu)\,e^{-i\phi} \quad .$$

Hence:

$$F_1^{-1}(x,t)\;\Phi_1^{-1}(\xi) = -\frac{3}{2}\cdot\frac{1}{4\pi}\cdot D\cdot\sin\theta\,(\cos\phi - i\sin\phi)\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) h(x,t)$$

In a similar way it can be found that:

$$F_1^{1}(x,t)\;\Phi_1^{1}(\xi) = -\frac{3}{2}\cdot\frac{1}{4\pi}\cdot D\cdot\sin\theta\,(\cos\phi + i\sin\phi)\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right) h(x,t)$$

$$F_1^0(x,t)\ \Phi_1^0(\xi) = -\ 3 \cdot \frac{1}{4\pi} \cdot D \cdot \cos\phi\ \frac{\partial h(x,t)}{\partial z} \tag{33}$$

Note that the two expressions in (32) are complex conjugates; so that, upon addition, the imaginary terms cancel. On adding together (31) to (33), equation (30) is obtained. Then using (5), equation (29) is obtained.

Equation (29) constitutes an effective means of verifying empirically whether a given light field satisfies the conditions (i) and (ii) for a diffusion approximation. All three radiometric concepts, N, h, and $\mathbf{H}$ in (29) are readily measurable in practice. Hence if an empirical radiance distribution comes to within an accepted interval of approximation of a cardioid of revolution, then the classical diffusion equation may be used to describe such a light field. We note a rather interesting near-confirmation of the steady state form of (29) in the case of heavily overcast skies. Empirical measurements reported in [186] show that the radiance of the underside of a heavy cloud overcast has essentially the form of (29), i.e., the cardioidal form.

Approaches via Higher Order Approximations

We pause in our description of the three main approaches to diffusion theory to place the discussion of the preceding paragraphs into perspective. We wish to show in particular how the classical diffusion equation (20) (or its equivalent form (7)) takes its place somewhere near the bottom of an infinitely high ladder of successively more detailed diffusion-type equations, each obtainable by following well-defined principles of modification, such as (i) and (ii) above, of the basic system (27) of Sec. 6.3.

In order to facilitate the classification of the various approaches possible via the system (27) of Sec. 6.3, let us write:

$$\text{"}\mathbf{F}_a\text{"} \quad \text{for} \quad (F_a^{-a},\ F_a^{-a+1},\ \ldots,\ F_a^{-1},\ F_a^0,\ F_a^1,\ \ldots,\ F_a^a)$$

Thus, e.g., "$\mathbf{F}_0$" denotes (F_0^0), "$\mathbf{F}_1$" denotes (F_1^{-1}, F_1^0, F_1^1), and so on. In other words F_a is a $(2a+1)$ component vector centered on the component F_0^0. When we say F_a is zero, we mean that each of its $2a+1$ components is zero. Further, when we write "$\partial\mathbf{F}_a/\partial t$" we shall mean $(\partial F_a^{-a}/\partial t, \ldots, \partial F_a^a/\partial t)$. In a similar way we can define $\mathbf{F}_{\eta,a}$.

Now the two principles (i) and (ii) used above to arrive at the classical diffusion equation (20) (or its equivalent (7)) may be recast into the following equivalent forms:

(i) (if $a > 1$, then $\mathbf{F}_a = 0$) and (if $a > 0$, then $\mathbf{F}_{\eta,a} = 0$).

(ii) if $a > 0$, then $\partial\mathbf{F}_a/\partial t = 0$.

This relatively succinct way of describing the modification of the system (22) of Sec. 6.3 may form the basis of classifying various diffusion processes. Thus in the following list, let the vectors $\mathbf{F}_a$, $\mathbf{F}_{\eta,a}$ and their derivatives appearing there be the only vectors not set equal to zero in the indicated approximation derived from (27) of Sec. 6.3. The symbol in the "process type" column to the left of the nonzero vectors is a succinct way of denoting the numerical classification of the approximation; some suggestive names for the approximations are given to the right of the vectors. Thus the approximation [1/0] is that giving rise to the classical scalar diffusion equation derived earlier by setting to zero all terms in (27) of Sec. 6.3 except those of $\mathbf{F}_0, \partial\mathbf{F}_1/\partial t$, $\mathbf{F}_1$, $\mathbf{F}_{\eta,0}$.

TABLE 1

A short list of diffusion processes

Process type	Nonzero terms in (27) of Sec. 6.3	Name of associated diffusion process
[0/1]	$\mathbf{F}_0$; $\mathbf{F}_{\eta,0}$	Equilibrium
[0/t]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_{\eta,0}$	Monotonic
[1/0]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$; $\mathbf{F}_{\eta,0}$	Scalar
[1/t]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$; $\partial\mathbf{F}_1/\partial t$; $\mathbf{F}_{\eta,1}$	Wave
[2/0]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$, $\partial\mathbf{F}_1/\partial t$; $\mathbf{F}_2$; $\mathbf{F}_{\eta,1}$	Tensor
[2/t]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$, $\partial\mathbf{F}_1/\partial t$; $\mathbf{F}_2$, $\partial\mathbf{F}_2/\partial t$; $\mathbf{F}_{\eta,2}$	Wave-tensor

The present classification of diffusion processes places two theories below the scalar diffusion theory ("below" in the sense of "less complex"). The first of these, the equilibrium diffusion theory, merely serves to describe the radiometric state of affairs in an equilibrium situation by means of the equation:

$$-\alpha F_0^0 + \sigma_0 F_0^0 + F_{\eta,0}^0 = 0$$

which may be written:

$$h(x,t) = \frac{h_\eta(x,t)}{a} \tag{34}$$

Thus (34) holds for a uniform, steady light field in equilibrium with its emission sources distributed throughout a medium X. The term h_η/a is reminiscent of Kirchhoff's law in radiometry, or of the equilibrium radiance N_q (see (2) of Sec. 4.3). A slightly more detailed description is given by the monotonic diffusion equation:

$$\frac{1}{v}\frac{\partial h}{\partial t} = -ah + h_\eta \quad . \tag{35}$$

Thus the diffusion process [0/t] described in (35) gives rise to a light field whose scalar irradiance h at a point generally grows or decays monotonically with time. The scalar diffusion process [1/0] was discussed in detail above.

We next encounter the processes [1/t], which is one step more accurate and complex than the classical diffusion process [1/0]. This new process is called the *wave diffusion process* by virtue of the fact that its associated equation (derived from (27) of Sec. 6.3 in the general manner illustrated for the case of [1/0]) is a wave equation of the form

$$\boxed{A\frac{\partial^2 h}{\partial t^2} + B\frac{\partial h}{\partial t} - D\nabla^2 h = -ah + h_\eta} \tag{36}$$

where we have written:

"A" for $3D/v^2$, "B" for $(1 + 3Da)/v$ (37),(38)

Comparing (36) with (7), we see that the process [1/t] adds the next higher derivative term to the equation for the process [1/0], plus slightly modifying the coefficients of the derivatives of the latter's equation. The physical processes corresponding to (36) and to (7) differ markedly: (36) describes a general damped wave-like process which propagates outward from any epicenter at the finite speed $v/\sqrt{3}$. Indeed, (36) is the well-known *telegrapher's equation*, which describes in another context the propagation of wave signals through a resistive wave-conducting medium. Equation (7), on the other hand, is the classical diffusion equation which describes a general monotonic decaying (or growing) diffusion process (with absorption and emission of the diffusing entities) propagating with infinite speed from a given epicenter. Equation (7) may be essentially obtained from (36) by letting v become so large that the second-derivative term in (36) becomes negligible, i.e., so that A is small compared to B.

The next higher diffusion process beyond wave diffusion is the process [2/0]. A new entity enters the picture here with $\mathbf{F}_2$. Whereas $\mathbf{F}_1$ describes the vectorial properties of the radiant flux (see the description of the vector irradiance $\mathbf{H}$ in terms of the components of $\mathbf{F}_1$, in (14)), $\mathbf{F}_2$ describes the tensorial properties of the radiant flux, properties very much like those described by the stress tensor in fluid dynamics.

Our present goal has essentially been reached; we have shown the place of the classical diffusion theory in the hierarchy of diffusion theories possible in radiative transfer theory. It is seen that the classical diffusion equation (7) is neither the beginning nor the end of the possibilities of

describing diffusive transport of photons in an optical medium. However, equation (7) is on the borderline between those theories which, on the one hand, are too crude to admit useful descriptions, and those which, on the other hand, are more accurate in their descriptive powers, but which are relatively complex and intractable in the light of current mathematical techniques. It is because of this convenient middling ground straddled by the diffusion equation (7) that it has been so popular with researchers looking for easily handled, reasonably accurate quantitative accounts of natural light fields. Some of the simple models arising from (7) will be considered in Sec. 6.6.

The Approach via Isotropic Scattering

The third and final main approach to diffusion theory we shall consider in this section is that via the assumption of the isotropic scattering property for an optical medium. The nature of this assumption is quite different from those used in the preceding two approaches. The earlier approaches, via Fick's law and via the spherical harmonic method, were gotten under way by first tampering with the directional structure of the light field, i.e., by reducing its awesome directional complexity to some relatively innocuous, mildly varying form (see, e.g., (29)) so that, for example, either Fick's law or the [1/0] process defined in Table 1 above could cope with the resultant weakened field. The nature of the assumption we shall adopt the present discussion is such that it leaves inviolate the intricate geometric structure of the radiance field; but in order to inculcate a semblance of manageability into the field, it is to be hypothesized that the volume scattering function σ is independent of ξ' and ξ throughout the medium. The resultant light field belonging to such a σ is a relatively tame analytic object by natural light field standards--so tame, in fact, that some quite elegant mathematical analyses of the classical mold can be employed to carry to completion the exact solution of the resulting equations for scalar irradiance. The associated theory is called *exact diffusion theory*. The "exactness" of the theory resides in its mathematical procedures, and not necessarily in its fidelity as a physical theory.

The manner in which we shall approach exact diffusion theory will be such as to show the necessity of the isotropic scattering assumption in the construction of the theory. By holding back the invocation of the isotropic scattering assumption until the last stage of the main analysis, it shall become quite clear that this is the essential physical concession made by an otherwise elegant, powerful theory which in principle is applicable to arbitrary (finite or infinite) inhomogeneous media with both internal and external sources.

To begin, let the optical medium X be of arbitrary spatial extent (in Fig. 6.3 it is shown as being finite), generally inhomogeneous, with arbitrary volume scattering function σ and volume scattering attenuation function α, and with arbitrary emission function N_η defined throughout X, and boundary radiance distribution N_0. For simplicity of exposition,

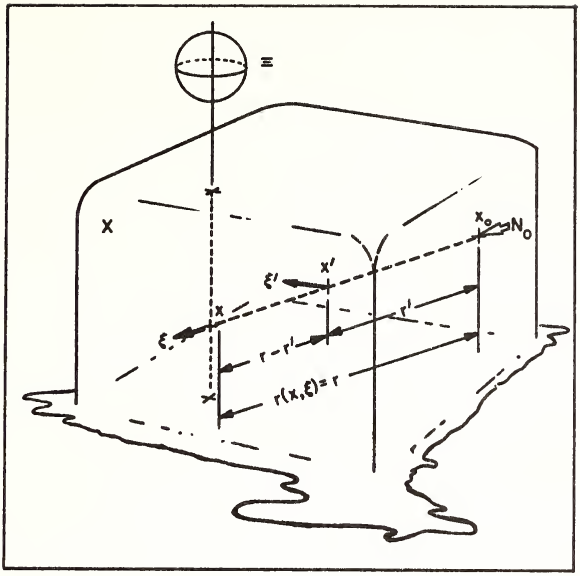

FIG. 6.3 Setting up the exact diffusion theory.

we postulate a steady-state radiance field N through X x Ξ. The corresponding formulation for the time-dependent field is obtained by simple modifications of the steady-state case. (See, e.g., (12) of 7.14.) The present discussion will be facilitated if at the outset we define certain integral operators. First, there is the path function operator **R** of Sec. 3.17:

$$\mathbf{R} = \int_{\Xi} [\,]\ \sigma(x;\xi';\xi)\ d\Omega(\xi') \quad .$$

The path radiance operator **T** of Sec. 3.17 will also be needed:

$$\mathbf{T} = \int_0^{r(x,\xi)} [\ \ ]\ T_{r-r'}(x'.\xi)\ dr' \quad .$$

The variables occurring in these operators are depicted in Fig. 6.3. Further, we shall write:

$$\text{"}\mathbf{U}\text{" for } \int_{\Xi} [\ \ ]\, d\Omega(\xi) \qquad (39)$$

This operator maps radiance distributions $N(x,\cdot)$ at a point x into their associated scalar irradiances $h(x)$, thus:*

$$h(x) = N\mathbf{U}(x) = \int_{\Xi} N(x,\xi)\, d\Omega(\xi) \tag{40}$$

or simply:

$$h = N\mathbf{U} = vu$$

for short, where vu is an alternate form of h (Sec. 2.7) involving radiant density u, and the speed of light, v. We shall also need the following two compositions of operators. First, the scattering operator $\mathbf{S}^1$ of Sec. 5.1:

$$\mathbf{S}^1 = \mathbf{RT}$$

and the composition $\mathbf{V}$, where we have written:

$$\text{"}\mathbf{V}\text{" for } \mathbf{TU} \tag{41}$$

The reader may verify directly from its definition that $\mathbf{V}$ has the representation:

$$\mathbf{V} = \int_X [\ ]\, K_\alpha(x',\cdot)\, dV(x') \tag{42}$$

which is the iteration of the integral operators $\mathbf{T}$ and $\mathbf{U}$, where for every x' and x in the medium we have written:

$$\text{"}K_\alpha(x',x)\text{" for } \frac{T_{r-r'}(x',\xi)}{|r-r'|^2} \tag{43}$$

and where $\xi = (x-x')/|r-r'|$; $|r-r'|$ is the distance $|x-x'|$ from point x' to point x as measured along the path of direction ξ. (As usual, "x" denotes a point of E_3, and as such is an ordered triple of real numbers.) The integration in $\mathbf{V}$ is with respect to the volume measure V. Thus $dV(x) = r^2 dr d\Omega(\xi)$, where $x = x_0 + r\,\xi$.

With all this machinery securely in place, we can go on to obtain the requisite equations so as to keep easily in view at all times the essential physical and mathematical features of the derivation.

The integral form of the equation of transfer ((2) of Sec. 3.15) with emission function N_η is:

---

*The notation "$N\mathbf{U}(x)$" denotes the value at x of the function $N\mathbf{U}$, and $N\mathbf{U}$ in turn is the result of operating on the function N with the operator $\mathbf{U}$.

$$N(x,\xi) = (N_o + N_\eta)\,\mathbf{T}(x,\xi) + N\mathbf{s}^1(x,\xi) \tag{44}$$

where* N_o is the initial radiance function within the medium due to boundary radiances, i.e., where we have written:

$$\text{"}N_o(x,\xi)\text{"} \quad \text{for} \quad N_o(x_o,\xi)\ \delta(x-x_o)$$

and where $N_o(x_o,\cdot)$ is the given incident radiance distribution at an arbitrary point x_o of X. By writing:

$$\text{"}N_\eta^o(x,\xi)\text{"} \quad \text{for} \quad (N_o + N_\eta)\ \mathbf{T}(x,\xi)$$

(44) becomes:

$$N(x,\xi) = N_\eta^o(x,\xi) + N\mathbf{s}^1(x,\xi)$$

Applying $\mathbf{U}$ to each side, we have

$$N\mathbf{U}(x) = N_\eta^o\mathbf{U}(x) + N\mathbf{s}^1\mathbf{U}(x)$$

whence:

$$h(x) = h_\eta^o(x) + (N\mathbf{R})\mathbf{TU}(x)$$

$$= h_\eta^o(x) + N_*\mathbf{TU}(x)$$

Hence

$$\boxed{h(x) = h_\eta^o(x) + N_*\mathbf{V}(x)} \tag{45}$$

where we have written:

$$\text{"}h_\eta^o(x)\text{"} \quad \text{for} \quad N_\eta^o\mathbf{U}(x) \tag{46}$$

Equation (45) is but one step away from being an integral equation for scalar irradiance h. On first sight it might appear promising to use the operator $\mathbf{U}$ on N_* to obtain the product of the volume total scattering function s(x) and scalar irradiance as follows:

$$N_*\mathbf{U}(x) = s(x)\ h(x)$$

Toward this end, the N_* term in (45) may have the identity operator I in the form of $\mathbf{UU}^{-1}$ slipped between N_* and $\mathbf{V}$, thus:

*The notation: "$(N_o + N_\eta)\mathbf{T}(x,\xi)$" denotes the value at (x,ξ) of the function $(N_o + N_\eta)\mathbf{T}$.

$$N_*\mathbf{U}\mathbf{U}^{-1}\mathbf{V}(x) = sh(\mathbf{U}^{-1}\mathbf{V})(x)$$

so that (45) could be written:

$$h(x) = h_\eta^o(x) + sh(\mathbf{U}^{-1}\mathbf{V})(x)$$

which is an operator equation in the unknown h. Unfortunately the inverse $\mathbf{U}^{-1}$ to the operator $\mathbf{U}$ does not generally exist, for the reason that there are many distinct radiance distributions at a point x giving rise to the same scalar irradiance h(x). This shows the necessity for assuming isotropic scattering for the medium if we are to obtain an integral equation for h. For then we have:

$$N_*(x,\xi) = N\mathbf{R}(x,\xi) = \frac{s(x)}{4\pi} h(x) \tag{47}$$

where we have assumed that:

$$s(x;\xi';\xi) = s(x)/4\pi \tag{48}$$

Using $N_*(x,\xi)$ in (45) as given by (47) we have:

$$\boxed{h(x) = h_\eta^o(x) + \frac{1}{4\pi}(hs)\,\mathbf{V}(x)} \tag{49}$$

This is the requisite general form of the basic equation of exact diffusion theory.

The natural solution of (49) is obtained by rearranging it as follows:

$$\begin{aligned} h_\eta^o(x) &= h(x) - \frac{1}{4\pi}(hs)\,\mathbf{V}(x) \\ &= h[I - \mathbf{V}_*](x) \end{aligned} \tag{50}$$

where we have written:

$$\text{"}\mathbf{V}_*\text{"} \quad \text{for} \quad \frac{1}{4\pi}\int_X [\ ]s(x')\,K_\alpha(x',\cdot)\,dV(x') \tag{51}$$

It is easily shown that the inverse $[I - \mathbf{V}_*]^{-1}$ of $I - \mathbf{V}_*$ generally exists, i.e., that $\mathbf{V}_*$ has the contraction property (cf. Sec. 5.14). Hence (44) yields:

$$\boxed{h(x) = h_\eta^o[I - \mathbf{V}_*]^{-1}(x)} \tag{52}$$

where generally:

$$[I - V_*]^{-1} = I + V_* + V_*^2 + V_*^3 + \dots \qquad (53)$$

Here V_*^2 is V_*V_*, i.e., the operator V_* followed by V^*. In general V_*^j is the operator V_*^{j-1} followed in application by V_*. This solution procedure is quite general. The operator V_*, which depends on the space X and its optical properties α and s, requires only the contraction property to be verified before it can be used in theory or practice.

An alternate form of (49), the form most often used in the classicial solution procedures, is obtained by rewriting (45) as:

$$h(x) = (N_o + N_\eta)\mathbf{TU}(x) + N_*\mathbf{V}(x)$$

$$= N_o\mathbf{V}(x) + (N_\eta + N_*)\mathbf{V}(x)$$

so that:

$$\boxed{h(x) = h^o(x) + (N_\eta + N_*)\ \mathbf{V}(x)} \qquad (54)$$

In order to obtain an equation in h only (all other terms being given functions) it follows, for the same reasons as those leading to (49), that the isotropic scattering assumption (48) must be adopted. In addition, if we are to retain the particular grouping of terms exhibited in (54), we may (though it is not strictly necessary to do so) also assume that N_η is of uniform directional structure, i.e., we assume:

$$N_\eta(x,\xi) = h_\eta(x)/4\pi \qquad (55)$$

where h_η is defined in (4). Under these conditions, (54) reduces to:

$$\boxed{h(x) = h^o(x) + \frac{1}{4\pi}\ (h_\eta + hs)\ \mathbf{V}(x)} \qquad (56)$$

If the space X is infinite in all directions about x, and α generally is not zero, then $h^o(x) = 0$, and (56) becomes:

$$\boxed{h(x) = \frac{1}{4\pi}\ (h_\eta + hs)\ \mathbf{V}(x)} \qquad (57)$$

which is the somewhat special but customary form of the integral equation on which the exact diffusion theory is based.

We now sketch the customary method of solution of (57). The medium is assumed homogeneous, so that $s(x)$ is independent of x and so that $K_\alpha(x',x)$ depends only on the difference $|x-x'|$. This assumption of homogeneity is necessary if the Fourier transform method (the usual method used) is to be applied to (57). Thus, if "$\mathcal{F}$" denotes the three-dimensional spatial Fourier transform operator for functions on X (which is now all of euclidean three space) we have, applying $\mathcal{F}$ to each side of (57):

$$(\mathcal{F}h)(k) = \frac{1}{4\pi} \mathcal{F}[(h_\eta + hs)\,\mathbf{V}]\,(k)$$

where k is the spatial frequency variable associated with the spatial variable x. The value of $\mathcal{F}[h]$ at k is written as "$\mathcal{F}[h;k]$", "$(\mathcal{F}h)(k)$", or "$\hat{h}(k)$", similarly with the inverse transform. Using the convolution theorem for Fourier transforms, (see, e.g., (6) of Sec. 7.14) this becomes:

$$\hat{h}(k) = \frac{1}{4\pi} (\hat{h}_\eta(k) + s\hat{h}(k))\,\hat{K}_\alpha(k) \qquad (58)$$

where for brevity we also write:

$$\text{"}\hat{K}_\alpha(k)\text{"} \quad \text{for} \quad \mathcal{F}[K_\alpha;k]$$

The carat over the letter "h" denotes, e.g., that h is the Fourier transform of h. The beauty and power of the Fourier transform method is now strikingly evident in (58): the integral operator equation (57) has been reduced to an algebraic equation in $\hat{h}(k)$ so that (58) may be directly solved for $\hat{h}(k)$:

$$\hat{h}(k) = \frac{\hat{h}_\eta(k)}{(4\pi - s\hat{K}_\alpha(k))}$$

Taking the inverse Fourier transform of each side, we have:

$$\boxed{h(x) = \mathcal{F}^{-1}\left[\frac{\hat{h}_\eta}{(4\pi - s\hat{K}_\alpha)}\right](x)} \qquad (59)$$

which rivals the natural solution (52) in simplicity and elegance (but evidently not in power and scope). The solutions of (57) will be discussed in more detail in Sec. 6.7.

The present discussion is concluded with the observation of how the radiance distribution $N(x,\cdot)$ is obtained from knowledge of scalar irradiance $h(x)$ when using exact diffusion theory. Once the scalar irradiance field h has been obtained from either (52) or (59), we use the representation of N_*, as given by (47), in the general relation (44):

$$N(x,\xi) = (N_o + N_\eta)\,\mathbf{T}(x,\xi) + N_*\mathbf{T}(x,\xi)$$

Thus:

Thus:

$$\boxed{N(x,\xi) = \left[N_o + N_\eta + \frac{hs}{4\pi}\right] \mathbf{T}(x,\xi)} \tag{60}$$

If the medium is source-free, so that $N_\eta = 0$, then

$$N(x,\xi) = \left[N_o + \frac{hs}{4\pi}\right] \mathbf{T}(x,\xi) \tag{61}$$

If the medium is in addition infinite, so that $N_o = 0$ at all interior points of X then

$$N(x,\xi) = \left[\frac{hs}{4\pi}\right] \mathbf{T}(x,\xi) \tag{62}$$

If the medium is also homogeneous, then

$$N(x,\xi) = (s/4\pi)\ [h\mathbf{T}(x,\xi)] \quad . \tag{63}$$

6.6 Solutions of the Classical Diffusion Equations

In this and the following section we shall exhibit some of the more useful general solutions of the classical and exact diffusion equations introduced in the preceding section. We begin with the classical diffusion equation in its simplest context.

Plane-Parallel Case

Consider an homogeneous plane-parallel source-free optical medium with a steady, stratified light field generated by incident flux at its upper boundary. For example, natural light fields in the seas, lakes, and harbors can supply such instances. Further instances may be found in heavy fogbanks and thick cloud layers. Suppose that the conditions for the diffusion equations hold in such media. What are the resultant forms of the light field--say the radiance distribution and associated scalar irradiance function--that the classical diffusion theory predicts for such media? We now seek the answers to these questions.

Starting with equation (7) of Sec. 6.5, and imposing the source-free, steady light field condition, we have:

$$D\,\nabla^2 h - ah = 0 \tag{1}$$

Recall that in a three-dimensional Cartesian coordinate system:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$

Since the light field is stratified, the x and y derivatives in $\nabla^2 h$ will be zero. Thus (1) reduces to:

$$D \frac{d^2h}{dz^2} - ah = 0 \quad . \tag{2}$$

Therefore, in its simplest guise, the classical diffusion equation (7) of Sec. 6.5 takes the form of a linear, second-order differential equation whose general solution for $a \neq 0$ is of the form:

$$h(z) = c_+ \, e^{\kappa z} + c_- \, e^{-\kappa z} \tag{3}$$

where we have written:

$$"\kappa" \quad \text{for} \quad \sqrt{\frac{a}{D}} \tag{4}$$

We call κ, as defined in (4), the (classical) *diffusion coefficient*. Recalling (27) of Sec. 6.5, we can express κ alternatively as:

$$\kappa = \sqrt{3a(\alpha - \bar{\mu}s)}$$

$$= \sqrt{3a(a + (1-\bar{\mu})s)}$$

The diffusion coefficient κ is the physical core of the solution (3) and, indeed, of all of the solutions of the classical diffusion equation. There may be variations in the geometry of a medium--spherically symmetric, cylindrically symmetric, plane parallel, as in the present case--and corresponding variations in the forms of solutions, as we shall see, but running through these cases, and common to them all, is the notion of the diffusion coefficient κ. Observe how κ depends jointly on the volume absorption coefficient a, the total volume scattering coefficient s, and on the mean cosine $\bar{\mu}$, which is a measure of the anisotropic scattering property of the medium.

As a special solution of (3) let the plane-parallel medium be infinitely deep, so that on physical grounds $c_+ = 0$ in (3) (see (12)). Then (3) can be shown to reduce to:

$$\boxed{h(z) = h(0)e^{-\kappa z}} \tag{6}$$

This is at once the most useful and representative example of the analytic form of light fields in natural optical media. The models for light fields in natural optical media come in all orders of complexity and power of representation, but in the final analysis all exhibit, in greater or lesser degree, and with an accuracy that generally increases with increasing depth, the overall *exponential* structure of natural light fields. The simplest of models of light fields in natural media--namely (2)--already exhibits this exponential structure

of the light fields. More sophisticated models will give correspondingly more detail on the structure of h(z) as a function of z; and still other models may sharpen the dependence of κ on a and s. Yet for all its simplicity, (2) has captured the salient analytic property of the light in natural hydrosols: that of exponentiality.

How does the magnitude of the diffusion coefficient compare with that of the volume attenuation coefficient α? We note first of all that these quantities are indeed comparable, both having dimensions of inverse length. From the representation (5) of κ we can build up the following chain of inequalities leading to α:

$$\kappa = \sqrt{3a(a + (1-\bar{\mu})s)} \leq \sqrt{3a(a+s)} \leq \sqrt{3(a+s)(a+s)} = \sqrt{3}\ \alpha \quad (7)$$

A more instructive inequality can be deduced provided that some explicit relation between s and a is hypothesized. Such a relation has already been observed in connection with the validity of diffusion theory. In the remarks following Fick's law (5) of Sec. 6.5, it was noted that the law holds when, among other things, the scattering-attenuation ratio ρ is at least 0.6. This condition on ρ in turn requires that: $s \geq (10/4)\ a > 2a$. It therefore seems reasonable to be able to use this inequality between s and a whenever diffusion theory itself is being used. Therefore, starting the chain of inequalities in (7) once again, we are now led to:

$$\kappa = \sqrt{3a(a + (1-\bar{\mu})s)} \leq \sqrt{3a(a+s)} < \sqrt{(a+s)(a+s)} = \alpha \quad .$$

Hence we see that, whenever diffusion theory is applicable, we must have:

$$\boxed{\kappa < \alpha} \quad (9)$$

The physical interpretation of (9) is clear: since κ is generally smaller than α, we have, depth-for-depth:

$$e^{-\alpha z} < e^{-\kappa z}$$

This means that transmitted radiant flux undergoing diffusion along a path of length z is greater at the end of the path than that having undergone pure attenuation. This may be seen also by direct appeal to the intuitive meaning of diffusion and attenuation in their technical senses used in transport theory: a stream of photons undergoing attenuation, loses photons under the joint action of absorption and scattering. Once a set of photons is scattered out of the beam, they are no longer considered part of the beam even though some of them may reenter the beam. A stream of photons undergoing diffusion, on the other hand, may scatter out of and back into the beam and be recounted upon rejoining the main stream. Thus the main loss mechanism for diffusion is absorption. Therefore, length for length, a packet of diffusing photons will have fewer loss casualties than a packet of attenuating (beam

transmitted) photons. This relation between κ and α may be alternatively stated by means of the attenuation length L_κ, where we have written:

$$"L_\kappa" \quad \text{for} \quad 1/\kappa \tag{10}$$

Then an equivalent statement to (9) is:

$$L_\alpha < L_\kappa \tag{11}$$

This inequality may be interpreted in a dual fashion to (9) as follows: The length of path in a medium over which a packet of photons undergoes a fixed fraction r of loss by means of diffusion is greater than the length of travel over which the packet undergoes the same fraction of loss by means of attenuation. In other words, a packet of diffusing photons will travel farther before incurring a given loss than it would travel before it incurred the same loss by pure attenuation.

If the plane-parallel medium is of finite depth d, then in general both c_+ and c_- in (3) are not zero. In fact c_+ and c_- are determined, for example, by specifying the scalar irradiances at any two depths in the medium. It is customary and convenient to specify $h(z)$ for $z = 0$ and $z = d$. Thus, supposing $h(0)$ and $h(d)$ given, we have from (3):

$$h(0) = c_+ + c_-$$

$$h(d) = c_+ e^{\kappa d} + c_- e^{-\kappa d}$$

We treat these two equations as linear algebraic equations in the unknowns c_+ and c_-, and find that:

$$c_\pm = \pm \frac{h(d) - h(0)\, e^{\mp\kappa d}}{e^{\kappa d} - e^{-\kappa d}} \tag{12}$$

We observe from these representations of c_+ and c_- that, for very deep media, $c_+ \cong 0$ and $c_- \cong h(0)$, so that in the limit of infinitely deep media, we return to the solution (6).

We consider next the specific form of the radiance distribution in the plane-parallel diffusion case. By (30) of Sec. 6.5 we know the general shape of the radiance distributions. But with a specific depth dependence of $h(z)$ now known, say in the case of (6) for an infinitely deep medium, the gradient of $h(z)$ is readily estimable, and so a specific estimate of $N(z,\xi)$ is possible. Since the light-field is stratified, we have

$$\nabla h = - \mathbf{k} \frac{dh(z)}{dz} \tag{13}$$

where $\mathbf{k}$ is the unit outward normal to the medium at its upper boundary. The medium has the standard terrestrially based

coordinate system for hydrologic optics (Sec. 2.4). Hence for infinitely deep media:

$$\boxed{N(z,\xi) = \frac{h(z)}{4\pi}\,[1 - 3\,\kappa\,D\xi \cdot \mathbf{k}]} \qquad (14)$$

where h(z) is given in (6). A similar formula for N(z,ξ) can be developed for finitely deep media using (3) with c_+ and c_- as given in (12).

Finally, we consider the upward and downward irradiances associated with the diffusion field in an infinitely deep optical medium. Using the ideas of Sec. 2.4 in which the properties of irradiance were described at length, let "H(z,+)" and "H(z,-)" denote the upward and downward irradiances in the medium. That is, in the terminology of (9), (10) of Sec. 2.4, we have written:

$$\text{"}H(z,+)\text{"} \quad \text{for} \quad H(z,\mathbf{k})$$

$$\text{"}H(z,-)\text{"} \quad \text{for} \quad H(z,-\mathbf{k}) \quad .$$

Then:

$$H(z,+) = \int_{\Xi_+} N(z,\xi)\ \xi \cdot \mathbf{k}\ d\Omega(\xi)$$

and:

$$H(z,-) = \int_{\Xi_-} N(z,\xi)\ \xi \cdot (-\mathbf{k})\ d\Omega(\xi)$$

which are based on (8) of Sec. 2.5. H(z,±) can be explicitly evaluated using (14) for N(z,ξ). Thus:

$$H(z,+) = \frac{h(z)}{4\pi} \int_{\Xi_+} (1 - 3\,\kappa\,D\xi \cdot \mathbf{k})\ \xi \cdot \mathbf{k}\ d\Omega(\xi)$$

$$= \frac{h(z)}{4}\,(1 - 2\kappa D) \qquad (15)$$

In a similar manner:

$$H(z,-) = \frac{h(z)}{4}\,(1 + 2\kappa d) \quad . \qquad (16)$$

From this we can estimate the ratio of downward to upward irradiance at each depth z in the medium. Writing:

$$\text{"}R(z,-)\text{"} \quad \text{for} \quad \frac{H(z,+)}{H(z,-)} \quad , \tag{17}$$

we have:

$$\boxed{R(z,-) = \frac{1 - 2\kappa D}{1 + 2\kappa D}} \tag{18}$$

for the *reflectance* $R(z,-)$ associated with an infinitely deep plane-parallel homogeneous medium as described by the concepts of classical diffusion theory. Observe that $R(z,-)$ in the present case is independent of z.

It is interesting to note that from (15), (16) and the concepts of vector irradiance (Sec. 2.8):

$$|\mathbf{H}(z)| = H(z,-) - H(z,+) = \kappa D h(z) \tag{19}$$

so that:

$$\boxed{\mathbf{H}(z) = -\kappa D h(z)\, \mathbf{k}} \tag{20}$$

Furthermore:

$$H(z,+) + H(z,-) = h(z)/2 \quad . \tag{21}$$

Relations (15) through (21) will be reconsidered in the light of the exact two-flow theory in plane-parallel media, as developed in Chapter 8.

Point Source Case

Consider an infinite homogeneous optical medium with an isotropic point source at the origin generating a steady light field throughout the medium. For example, a bright flare of uniform directional output deep in the ocean far from surface and bottom effects would generate such a light field. Flares deep within foggy atmospheric media such as in fogbanks and clouds also offer real instances of the present case. the plane-parallel case, we are interested in the scalar irradiance field and the radiance field set up by the point as in source in the surrounding medium. In particular, we now study these fields as predicted by classical diffusion theory.

At all points of the medium other than at the position of the point source, equation (7) of Sec. 6.5 governs the resultant scalar irradiance field:

$$D\nabla^2 h - ah = 0 \tag{22}$$

The appropriate coordinate frame at present would be a spherical polar coordinate frame with origin at the point source. For then $\nabla^2 h$ takes a particularly simple form because of the spherical symmetry of the field about the point source. Thus, in general for spherical coordinates in which $x = (r,\theta,\phi)$:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial\phi^2} \tag{23}$$

By spherical symmetry we now have:

$$\nabla^2 h = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dh}{dr}\right)$$

$$= \frac{d^2 h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{1}{r}\frac{d^2(rh)}{dr^2} \tag{24}$$

Hence (22) becomes in the present case:

$$D \frac{1}{r}\frac{d^2(rh)}{dr^2} - ah = 0 \tag{25}$$

If we write, *ad hoc:*

$$\text{"F"} \quad \text{for} \quad rh \quad , \tag{26}$$

then (25) becomes:

$$D \frac{d^2 F}{dr^2} - aF = 0 \tag{27}$$

and we are back, mathematically, to the case described by (2). Hence the general solution of (27) is:

$$F(r) = c_+ e^{\kappa r} + c_- e^{-\kappa r} \quad , \tag{28}$$

or, in view of (26):

$$h(r) = \frac{1}{r}(c_+ e^{\kappa r} + c_- e^{-\kappa r}) \tag{29}$$

In view of the spherical symmetry, the values of $h(x)$ depend only on r, where $x = (r,\theta,\phi)$, and we therefore have written for brevity "$h(r)$" instead of "$h(x)$".

For the presently considered setting, namely an infinite medium, we can, for physical reasons, immediately set c_+ to zero. The exact mathematical procedure for this is completely analogous to that used to obtain (12). Therefore the scalar irradiance about a point source generally behaves in the manner described by the following equation:

$$h(r) = \frac{c_- \, e^{-\kappa r}}{r} \tag{30}$$

That is to say, h(r) falls off jointly as the inverse first power of r and exponentially with r. The constant c_- can be evaluated if we use the connection between vector irradiance **H** and scalar irradiance h given in Fick's law (5) of Sec. 6.5:

$$\begin{aligned} |\mathbf{H}(r)| &= |D\nabla h(r)| \\ &= |D \frac{dh(r)}{dr}| \\ &= \frac{Dc_- \, e^{-\kappa r}}{r^2} [1 + \kappa r] \end{aligned} \tag{31}$$

Here we have used the fact that **H**(r) is directed radially outward from the source (again a consequence of spherical symmetry). The magnitude of **H**(r) is the net outward irradiance at each point of a spherical surface of radius r. Hence:

$$4\pi r^2 \; |H(r)| = 4\pi D c_- \; e^{-\kappa r} \; [1 + \kappa r] \tag{32}$$

is the total net outward radiant flux, call it "P_r", across the spherical surface of radius r. For general radii r we do not know *a priori* the magnitude of this net outward flow. Even if we knew the radiant flux output, say P_o, of the point source at the origin, there is no *a priori* connection between P_o and P_r. However, if one measures P_r for some r, then (27) yields up at once an empirical estimate of c_-. On further examination of (27) it appears possible to devise a theoretical means of finding c_- by considering P_r for very small values of r. In such cases the spherical volume enclosing the point source is so small that the *net* outward flow across the boundary due to the field flux is zero, or very nearly so, for the reason that there is very small chance for a packet of photons diffusing into and then out of the spherical volume to lose any members by absorption during the traversal of the volume (the main loss mechanism which affects diffusing particles). At any rate, it is clear *a priori* that this chance goes to zero in magnitude with the radius of the sphere. Hence in the limit of zero radius the net outward flow across the spherical surface is due solely to the point source's output P_o. Thus from (32) we find:

$$\begin{aligned} P_o &= \lim_{r\to o} P_r = \lim_{r\to o} 4\pi D c_- \; e^{-\kappa r} \; [1 + \kappa r] \\ &= 4\pi D c_- \quad , \end{aligned}$$

whence:

$$\boxed{h(r) = \frac{P_o \, e^{-\kappa r}}{4\pi D r}} \tag{34}$$

Equation (34) describes the scalar irradiance at distance r from a point source of isotropic radiant flux output P_o. The flux is evolving in a diffusing medium with diffusion constant D, and diffusion coefficient κ. Equation (34) may be phrased in terms of the radiant intensity J_o of the point source. Thus, using (17) of Sec. 2.9,

$$\boxed{h(r) = \frac{J_o\, e^{-\kappa r}}{Dr}} \tag{35}$$

where we have written:

$$"J_o" \quad \text{for} \quad \frac{P_o}{4\pi} \tag{36}$$

The radiance distribution associated with the point source diffusion problem is obtained at each distance r from the point source by means of (30) of Sec. 6.5, now using as gradient:

$$\nabla h(x) = - \mathbf{r}\, \frac{dh(r)}{dr} \quad , \tag{37}$$

where $\mathbf{r}$ is the unit radial vector directed *toward* the point source. The gradient (37) was evaluated in (31), so that with the aid of (34):

$$\nabla h(x) = \frac{P_o\, e^{-\kappa r}\, (1+ \kappa r)}{4\pi D r^2}\, \mathbf{r} \tag{38}$$

Hence

$$\boxed{\mathbf{H}(r) = - D\, \nabla h = - \frac{Dh(r)(1 + \kappa r)}{r}\, \mathbf{r}} \tag{39}$$

Therefore, by means of (30) of Sec. 6.5 we have:

$$\boxed{N(r,\xi) = \frac{h(r)}{\pi} \left[1 - 3\, \frac{D(1 + \kappa r)}{r}\, \xi \cdot \mathbf{r} \right]} \tag{40}$$

where h(r) is given in (34). Equation (40) represents the radiance function in an infinite medium with isotropic point source under the usual conditions for classical diffusion theory (see process [1/0], Table 1, Sec. 6.5). A similar formula can be developed for finite spherical media. However, in this case care must be taken to see that the basic diffusion conditions hold, in particular so that Fick's law (5) of Sec. 6.5 is applicable. Observe that at great distances r from the source, the expression for $N(r,\xi)$ as given in (40) approaches (14) of the plane-parallel case. Thus the radiance distribution at great distances from the point source

settles down to become the product of a spatial factor and a directional factor. In other words, the spatial and directional dependences of $N(r,\xi)$ eventually multiplicatively uncouple at great distances from the point source. This fact was used as a motivation for the spherical harmonic method in Sec. 6.1, and will be discussed in Sec. 10.6 as a special case of the general asymptotic radiance theorem (Sec. 10.5).

We conclude the discussion of the point source case by deriving the expressions for the outward and inward irradiances $H(r,\pm)$, where we have written:

$$"H(r,\pm)" \quad \text{for} \quad H(r,\pm\mathbf{r}) \tag{41}$$

on the basis of the general irradiance (11) of Sec. 2.6. Thus, in a manner similar to that used to find (15) and (16), we have for the point source context:

$$H(r,\pm) = \frac{h(r)}{4\pi}\left[1 \mp \frac{2D(1+\kappa r)}{r}\right] \tag{42}$$

so that, analogously to (18), we have:

$$R(r,-) = \frac{1 - [2D(1+\kappa r)]/r}{1 + [2D(1+\kappa r)]/r} \tag{43}$$

for the reflectance $R(r,-)$ of the medium at distance r from the point source, where we have written:

$$"R(r,-)" \quad \text{for} \quad \frac{H(r,+)}{H(r,-)} \tag{44}$$

Unlike the reflectance $R(z,-)$ obtained in the plane-parallel case, the present reflectance $R(r,-)$ varies with the distance r. In the limit of increasing r, however, $R(r,-)$ approaches the form of $R(z,-)$. Observe also how the values of r cannot be arbitrarily small and still have formulas such as (40) and (43) physically meaningful. The reason for this breakdown of the diffusion theory formulas is traceable to the eventual inapplicability of the original Fick's law hypothesis. In the presence of the highly varying directional structure of radiance distributions that occur near point sources, the simple cardioidal structure of radiance distributions, characteristic of diffusion theory, simply does not hold. It is at this point that the spherical harmonic approach to diffusion theory, on which the cardioidal radiance law is based, shows the inapplicability of Fick's law assumption. See, e.g., (14), (20), and (29) of Sec. 6.5.

Discrete Source Case

We take up once again the setting described in the point source case, just concluded. Now we imagine a set of point sources distributed throughout the infinite homogeneous medium. This set may be finite or infinite. In either case we assume the "points" to be disjoint, small regular-shaped volumes of given minimum size, the centers at points x_j. The definition of point source adopted in the present case is

that given in Sec. 2.9. Our present purpose is to derive the equations for the scalar irradiance and vector irradiance fields associated with such sets of point sources. From these representations, the radiance field follows at once using (29) of Sec. 6.5.

Suppose the set of point sources is located at the points x_1, x_2, ..., in the medium and that point x_j has isotropic radiant flux output $P_0(x_j)$. It follows from (34) and the interaction principle (which now assures superimposability of effects) that the total irradiance $h(x)$ generated at x by the point sources at each x_j is given by:

$$\boxed{h(x) = \sum_{j=1}^{\infty} \frac{P_0(x_j)\ e^{-\kappa|x-x_j|}}{4\pi\ D|x-x_j|}} \qquad (45)$$

where, as usual, "$|x-x_j|$" denotes the distance between point x and point x_j. In case only a finite number n of point sources are present, we set $P_0(x_j) = 0$ in (45) for every j such that $j > n$. There is no question about the convergence of the infinite series in (45) since we have assumed that each x_j is embedded in a small but finite volume of given minimum size. Hence the points x_j cannot all cluster in any finite region of space. The exponential factors in (45) then assure convergence of the infinite series, since the distances $|x-x_j|$ increase regularly with j, in the limit.

The relation (45) has a deceptive amount of generality. We could, if required, partition all of euclidean three space (except some arbitrarily small neighborhood of x) into cubes of varying sizes if need be. Then each cube with center x_j is assigned an output $P_0(x_j)$. Equation (45) then gives the total scalar irradiance at x generated by these discrete sources throughout space.

As an example of the preceding observation, suppose that small, finite, contiguous volumes are used to similate a thin cylindrical region with a straight-line segment in space as axis and along which sources are distributed. Such cylinders may simulate narrow beams of radiant flux sent out by highly directional sources, for example laser sources. In this case $P_0(x_j)$ is generated by the scattering, within the jth volume segment, of the residual flux of the beam reaching the jth volume. Thus, suppose a laser source is at point x_0 and directed along the path $P_r(x_0,\xi)$ with initial point x_0 and direction ξ, as in Fig. 6.4. Partition the beam, which has initial radiance N_0, into n parts, each a cylinder of length r/n and initial point x_j (= x_0 + (jr/n)). Finally, suppose the volume scattering function σ is independent of ξ', ξ, i.e., that isotropic scattering prevails throughout the medium. Then it is clear that:

$$N_0\ e^{-jr\alpha/n}$$

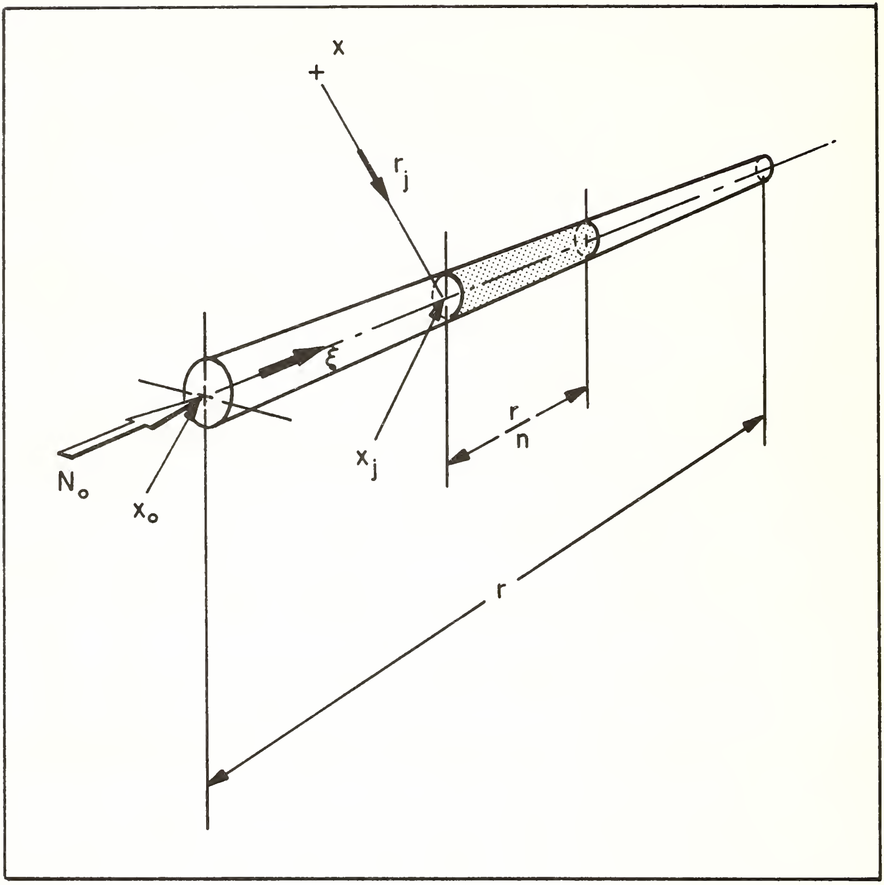

FIG. 6.4 Geometry for a narrow cylindrical beam source of radiant flux in diffusion theory.

is the residual radiance reaching the initial point x_j of the jth cylindrical part of the beam. From this and the definition of path function it follows that:

$$N_o \, e^{-jr\alpha/n} \, (s/4\pi)$$

is the path function value at the initial point x_j of the jth cylindrical part of the beam. Because scattering is isotropic, this value is assigned to each direction about x_j. Since path function values have the dimension of intensity per unit volume (e.g., see note (h) for Table 3 in Sec. 2.12), we can make the following assignation: To

$$P(x_i)/4\pi \; (= J(x_i))$$

in (45), we assign:

$$N_o\, e^{-jr\alpha/n}\, (sV(x_j)/4\pi) \quad ,$$

where $V(x_j)$ is the volume of the jth part of the beam, so that (45) now becomes:

$$h(x) = \frac{N_o s}{4\pi D} \sum_{j=1}^{n} \frac{e^{-jr\alpha/n}\, e^{-\kappa|x-x_j|}\, V(x_j)}{|x-x_j|} \tag{46}$$

This shows how the discrete-source case can simulate important internal source problems in natural optical media, provided, of course, that the basic diffusion point source model is valid for the given medium.

The radiance distribution associated with a discrete source scalar irradiance field given by (45) is obtained by appeal to the interaction principle, so that by simply adding together terms of the form shown in (40), the desired radiance distribution is obtained. An alternate representation of $N(x,\xi)$ is obtainable as follows: From (39) and the interaction principle it is clear that the vector irradiance generated by the point sources at x_1, x_2, ... is:

$$\mathbf{H}(x) = - \sum_{j=1}^{\infty} \frac{D\, h_j(x)\, (1+\kappa|x-x_j|)}{|x-x_j|}\, \mathbf{r}_j \tag{47}$$

where we have written:

$$\text{"}h_j(x)\text{"} \quad \text{for} \quad \frac{P_o(x_i)\, e^{-\kappa|x-x_j|}}{4\pi D|x-x_j|}$$

and where $\mathbf{r}_j$ is the unit vector directed from the observation point x to the jth source point x_j (see Fig. 6.4). Then using $\mathbf{H}(x)$ and $h(x)$ as given, respectively, by (47) and (45), the radiance $N(x,\xi)$ at x in the direction ξ is given once again by (29) of Sec. 6.5.

Continuous Source Case

We now make the transition from the discrete source case, just concluded, to the continuous source case. We begin with the finite version of (45) in which we have partitioned a subset X_n of the infinite medium into a set of n small volumes X_j ("small" in the sense of less than one attenuation length in diameter) each of which has a radiant flux output of $P_o(x_j)$, where x_j is a point of X_j. Hence the radiant flux output per unit volume about x_j is very nearly $P_o(x_j)/V(X_j)$, where $V(x_j)$ is the volume of X_j. We assume that the radiant flux output of X_j is uniform in all directions about x_j. Then the radiant intensity per unit volume:

$$\frac{P_o(x_j)}{4\pi\ V(X_j)}$$

may be represented by an emission radiance distribution $N_\eta(x_j,\xi)$ which is independent of direction ξ. (Recall that N_η has the same dimensions as path function N_*, and that the latter's dimensions may be characterized as radiant intensity per unit volume). Therefore, using the definition of h_η in (4) of Sec. 6.5, we may write:

$$\frac{P_o(x_j)}{4\pi\ V(x_j)} = N_\eta(x_j,\xi) = h_\eta(x_j)/4\pi \quad , \tag{48}$$

so that:

$$h_\eta(x_j) = \frac{P_o(x_j)}{V(X_j)} \quad . \tag{49}$$

With this meaning of $h_\eta(x_j)$, the finite version of (45) may be rewritten as:

$$h(x) = \sum_{j=1}^{n} \frac{h_\eta(x_j)\ e^{-\kappa|x-x_j|}\ V(x_j)}{4\pi D|x-x_j|} \tag{50}$$

By letting the partition of X_η become finer, so that in the limit the associated Riemann integral over X_η is obtained, (50) becomes:

$$\boxed{h(x) = \int_{X_\eta} \frac{h_\eta(x')\ e^{-\kappa|x-x'|}}{4\pi D|x-x'|}\, dV(x')} \tag{51}$$

This is the desired representation of the scalar irradiance $h(x)$ generated by isotropic point sources of strength $h_\eta(x)$ watts per unit volume, at points x' throughout a region X_η of the medium X. In analogy to (43) of Sec. 6.5 we write:

$$\text{"}K_\kappa(x',x)\text{"} \quad \text{for} \quad \frac{e^{-\kappa|x-x'|}}{4\pi D|x-x'|} \tag{52}$$

and

$$\text{"}\mathbf{W}\text{"} \quad \text{for} \quad \int_{X_\eta} [\ ]\ K_\kappa(x',x)\ dV(x') \tag{53}$$

so that (51) may be written:

$$\boxed{h(x) = \int_{X_\eta} h_\eta(x')\ K_\kappa(x',x)\ dV(x') = h_\eta \mathbf{W}(x)} \qquad (54)$$

Finally, the vector irradiance $\mathbf{H}(x)$ in the continuous source case can be obtained by starting with (47) and going to the Riemann integral counterpart of that sum. Thus, suppose initially the sum is finite and that the sources are confined to a part X_η of the medium. Then, as before the set X_η is partitioned and "$h_\eta(x_j)$" introduced to denote the unit volume output of the medium at point x_j in X_η. Thus (47) becomes:

$$\mathbf{H}(x) = \sum_{j=1}^{n} \frac{D\ h_\eta(x_j)\ K_\kappa(x_j,x)(1+\kappa|x-x_j|)(-\boldsymbol{r}_j)\ V(x_j)}{4\pi|x-x_j|}$$

in which (52) is used. Observe that $-\boldsymbol{r}_j$ is $(x-x_j)/|x-x_j|$ so that as the partition of X_η is made suitably fine, the sum has the limit:

$$\boxed{\mathbf{H}(x) = \int_{X_\eta} \frac{D\ h_\eta(x')\ K_\kappa(x',x)(1+\kappa|x-x'|)}{4\pi|x-x'|^2}\ (x-x')\ dV(x')} \qquad (55)$$

When h(x) and H(x), as given by (51) and (55), are used in (29) of Sec. 6.5, we obtain the appropriate radiance function for the diffusing light field generated by a continuous distribution of sources in X_η. The limitations of the point source case are as considered above. Indeed, since the point source case fails for points of observation too near the point source, it follows that points of observation x in (51) and (55) should not be in X_η, and preferably at some distances from X_η. We must impose this limitation on all diffusion integrals in practice. This problem of the proximity of the sources of the diffusing field will be examined in the following paragraphs.

Primary Scattered Flux as Source Flux

Time and again in the preceding illustrations of the diffusion method, precautionary observations were required on the use of the various derived equations because of possible inapplicability of Fick's law. For example, when an observation point x is too near a point source point x_0 in an otherwise suitably diffusing medium, the radiance distribution

about x may depart too markedly from the cardioidal distribution indigenous to classical diffusion theory. This departure is due principally to the highly directional residual radiance originating at x_0 and arriving at x. It would therefore seem desirable to improve the radiometric conditions prior to applying the classical diffusion theory by first computing the primary scattered radiance field generated by the given sources and using this radiance field as the source field in the continuous diffusion case considered above. We shall explore this possibility and its generalization in this and the subsequent paragraph.

In order to correctly implement the present discussion it seems best to return directly to the basic equation of transfer for scalar irradiance, (1) of Sec. 6.5. Our immediate task is to decompose the steady-state scalar irradiance h(x) into its residual component h^o and its diffuse component h^*, where the basis for these concepts were defined in (15) and (22) of Sec. 5.1. Thus, using the operator **U** in (39) of Sec. 6.5, we write:

$$\text{"}h^*(x)\text{"} \quad \text{for} \quad N^*(x,\cdot)\,\mathbf{U} \tag{56}$$

so that:

$$h(x) = h^o(x) + h^*(x) \tag{57}$$

and

$$h^*(x) = \sum_{j=1}^{\infty} h^j(x) \quad . \tag{58}$$

In other words, the scalar irradiance h(x) consists of the sum of all scalar irradiances $h^n(x)$ associated with n-ary radiance distributions $N^n(x,\cdot)$ at x. Hence $h^*(x)$ consists of radiant flux having undergone one or more scattering operations. Clearly, (57) may be obtained immediately from (4) of Sec. 5.4 by applying the operator **U** (cf. (39) of Sec. 6.5). That is, from

$$N = N^o + N^* \tag{59}$$

we obtain

$$N\mathbf{U} = (N^o + N^*)\,\mathbf{U} = N^o\mathbf{U} + N^*\mathbf{U}$$

that is:

$$h = h^o + h^* \tag{60}$$

We now use this mode of decomposition of h in the steady state version of (1) of Sec. 6.5. The details are as follows, starting with:

$$\xi \cdot \nabla N = -\alpha N + \int_{\Xi} N\sigma\, d\Omega + N_\eta$$

we first decompose N as in (59) to obtain:

$$\xi \cdot \nabla (N^o + N^*) = -\alpha(N^o + N^*) + \int_\Xi (N^o + N^*)\, \sigma d\Omega + N_\eta$$

Hence:

$$\xi \cdot \nabla N^* = -\alpha N^* + \int_\Xi N^* \sigma d\Omega + \int_\Xi N^o \sigma d\Omega \tag{61}$$

where we have used the relation:

$$\xi \cdot \nabla N^o = -\alpha N^o + N_\eta$$

which follows from (2) of Sec. 5.8. Recalling the definition of N_*^1 ((2) of Sec. 5.1), (61) can be cast into the form:

$$\boxed{\xi \cdot \nabla N^* = -\alpha N^* + \int_\Xi N^* \sigma d\Omega + N_*^1} \tag{62}$$

This is the equation of transfer which governs the diffuse radiance field N* consisting of primary and higher order scattered flux. An alternate derivation of (62) was performed in (7) of Sec. 5.2. The source for the field N* is the first order path function N_*^1. Because the residual radiance N^o coming in from the boundaries of the medium, and emission radiance N_η are now absent from N*, the directional structure of N* is considerably milder than that of N_1 so that Fick's law is more likely to hold for N* than N.

It is to the scalar irradiance h* induced by N* that we now direct attention and derive from (62) the required diffusion equation for h*. Thus, applying the operator **U** to (62) we have:

$$\nabla \cdot \mathbf{H}^* = -ah^* + h_*^1 \tag{63}$$

where:

$$h_*^1 = h^o s$$

and where we write:

$$\text{"}\mathbf{H}^*\text{"} \quad \text{for} \quad \int_\Xi N^* \xi \, d\Omega \tag{64}$$

Assuming Fick's law to hold between H* and h* (cf. (5) of Sec. 6.5), i.e., assuming:

$$\boxed{\mathbf{H}^* = -D\,\nabla h^*} \tag{65}$$

(63) becomes:

$$\boxed{-D\,\nabla h^* + ah^* = h_*^1} \tag{66}$$

This is the requisite steady-state diffusion equation for h^* in which the primary scattered scalar irradiance h_*^1 serves as an auxiliary source to the basic emission sources h_η in the medium. The assumption of Fick's law for h^* in (65) has a better chance of being valid than for h, since h has h^0 as a component which can be associated with highly directional flows from boundaries and internal sources.

The theory of the continuous source developed above and summarized in (51) and (55) may now be applied to the case where h_η in those equations is replaced by h_*^1. The proof of this procedure is based on the fact that the derivation of (51) and (55) ultimately rests on the steady-state version of (7) of Sec. 6.5; and this has just been shown to be identical with (66) in which h_η in the earlier equation is now replaced by h_*^1.

We now illustrate the use of (66) by means of a simple example. We consider an isotropic point source in an infinite homogeneous medium which scatters isotropically (i.e., is independent of ξ' and ξ). The source is at the origin and in reality constitutes a very small, essentially transparent sphere of radius r_0 which has a uniform surface radiance N_0. Thus the radiant emittance of the spherical surface is πN_0 and therefore the total flux output is $4\pi^2 r_0^2 N_0$. The average flux per unit volume of the spherical source is $4\pi^2 r_0^2 N_0/(4\pi r_0^3/3) = 3\pi N_0/r_0$. It is this output which would customarily be used in the estimate of h_η in the continuous case (cf. (49)). However, now the source is allowed first to generate a primary scattered flux field h_*^1 in the space surrounding it. In principle this primary scattered flux is generated at every point of the medium and may be estimated as follows at a point x' a distance $r' > r_0$ from the center of the spherical source. First note that $r' = |x'|$. Then let $\Omega(|x'|)$ $(=\Omega(r'))$ be the magnitude of the solid angle subtended by the sphere at vantage point x'. Then very nearly:

$$N_*^1(x',\xi) = \int_\Xi N^0 \sigma(x';\xi';\xi)\, d\Omega(\xi') = N^0 \Omega(r') s/4\pi = N_0\, e^{-\alpha r'}\Omega(r') s/4\pi$$

for every ξ. Hence:

$$h_*^1(x') = \int_{\Xi} N_*^1(x',\xi)\, d\Omega(\xi)$$

$$= N_o\, e^{-\alpha r'}\Omega(r')s \quad . \tag{67}$$

This representation is not exact because the integration over the set of directions from the emitting sphere assumed the distances from the point x' to the various points on the spherical surface were all equal to the fixed distance r'. However (67) should give excellent estimates of $h_*^1(x')$ for points x' when the sphere is viewed as a point source. We shall adopt (67) as a working basis in the present example.

We now use equation (51) with $h_\eta(x')$ in that equation replaced by $h_*^1(x')$ as given in (67). Here r' is the distance from x' to the origin; hence $r' = |x'|$. With these observations (51) now lets us write:

$$h^*(x) = N_o s \int_X \frac{e^{-[\kappa|x-x'| + \alpha|x'|]}\, \Omega(|x'|)\, dV(x')}{4\pi D|x-x'|} \tag{68}$$

Finally, the residual scalar irradiance $h^o(x)$, was essentially evaluated in arriving at (67); that is, the scalar irradiance induced by the small sphere is:

$$h^o(x) = N_o\, e^{-\alpha|x|}\, \Omega(|x|) \quad . \tag{69}$$

The full scalar irradiance h(x) for the present problem is, according to (57), the sum of $h^o(x)$ and $h^*(x)$ as they are given in (68) and (69). A generalization of (68) is readily effected by letting N^o vary in direction. All this means formally is that "N^o" goes under the integral sign in (68). In this case, the approximation of $h^o(x)$ by $N_o\Omega(|x|)\, e^{-\alpha|x|}$ must be examined. This will not be attempted here.

Higher Order Scattered Flux as Source Flux

The preceding example of the use of primary scattered radiant flux as source flux in the classical diffusion equation seems sufficiently useful to encourage carrying out the underlying idea of the example to its logical conclusion. Toward this end, suppose that it is possible to compute the first n+1 scattering orders for radiance: N^j, j = 0, 1, ..., n. We then supplement this exact calculation by estimating the radiance function

$$\sum_{j=n+1}^{\infty} N^j$$

using diffusion theory. Clearly this procedure includes that of the preceding discussion as a special case; in fact it is the case $n = 0$.

As in the special investigation for the case $n = 0$, we begin with the steady-state equation of transfer:

$$\xi \cdot \nabla N = - \alpha N + \int_{\Xi} N \sigma d\Omega + N_\eta$$

and now write N as:

$$N = \sum_{j=0}^{\infty} N^j = \sum_{j=0}^{n} N^j + \sum_{j=n+1}^{\infty} N^j$$

$$= N^{(n)} + N^{(n,*)} \tag{70}$$

where the definitions of the two terms $N^{(n)}$ and $N^{(n,*)}$ are implicit in (70). Thus in particular $N^{(o)} = N^o$ and $N^{(o,*)} = N^*$.

Using this decomposition in the equation of transfer, we have:

$$\xi \cdot \nabla(N^{(n)} + N^{(n,*)}) = -\alpha(N^{(n)} + N^{(n,*)}) + \int_{\Xi} (N^{(n)} + N^{(n,*)}) \sigma d\Omega + N_\eta \tag{71}$$

Now, from (1) of Sec. 5.2 we have for every $j \geq 1$

$$\xi \cdot \nabla N^j = -\alpha N^j + \int_{\Xi} N^{j-1} \sigma d\Omega \tag{72}$$

and from (2) of Sec. 5.8:

$$\xi \cdot \nabla N^o = -\alpha N^o + N_\eta \tag{73}$$

By adding equations (72) and (73) together from $j = 1$ up to $j = n$, we obtain:

$$\xi \cdot \nabla N^{(n)} = -\alpha N^{(n)} + \int_{\Xi} N^{(n-1)} \sigma d\Omega + N_\eta \tag{74}$$

This equation is now used with (71) to reduce the latter to:

$$\xi \cdot \nabla N^{(n,*)} = -\alpha N^{(n,*)} + \int_{\Xi} N^{(n,*)} \sigma \, d\Omega + N_*^{n+1} \tag{75}$$

This equation is the direct generalization of (62), the latter being obtained by setting n = 0 in (75).

Next the operator **U** ((39) of Sec. 6.5) is applied to each side of (75); the result is:

$$\nabla \cdot \mathbf{H}^{(n,*)} = -a \, h^{(n,*)} + h_*^{n+1} \tag{76}$$

The final step is to hypothesize that Fick's law holds between $\mathbf{H}^{(n,*)}$ and $h^{(n,*)}$:

$$\mathbf{H}^{(n,*)} = -D\nabla h^{(n,*)} \tag{77}$$

so that (76) becomes:

$$\boxed{-D \nabla h^{(n,*)} + ah^{(n,*)} = h_*^{n+1}} \tag{78}$$

This is the requisite diffusion equation for $h^{(n,*)}$. It is a direct generalization of (66) which is the case n = 0. The source term for the flux $h^{(n,*)}$ is (n+1)-ary scattered flux, which should have relatively mild direction structure, so that (77) has a good chance of holding in practice. In general, the greater the n, the more likely--on intuitive grounds--(77) would seem to hold. (See the discussion following (13) of Sec. 5.12.)

Once $h^{(n,*)}$ is obtained by solving (78) with the continuous source h_*^{n+1}, using, e.g., (51) with h_n replaced by h_*^{n+1}, we then find the complete scalar irradiance h by noting that

$$\boxed{h(x) = h^{(n)}(x) + h^{(n,*)}(x)} \quad . \tag{79}$$

where we write:

$$\text{"}h^{(n)}\text{"} \quad \text{for} \quad N^{(n)} \, \mathbf{U} \tag{80}$$

and:

$$\text{"}h^{(n,*)} \quad \text{for} \quad N^{(n,*)} \, \mathbf{U} \tag{81}$$

From $h^{(n,*)}(x)$ we can then find $\mathbf{H}^{(n,*)}(x)$ using (77) and so, in turn, $N^{(n,*)}(x,\xi)$ using the diffusion equation

(29) of Sec. 6.5 as a model. This diffusion-based estimate of $N^{(n,*)}(x,\xi)$ is then added to the known radiance $N^{(n)}(x,\xi)$.

Time-Dependent Diffusion Problems

Time-dependent radiative transfer problems arise, for example, whenever extremely short pulses of radiant energy are released in scattering-absorbing media, and when the evolution of the subsequent scattered radiant energy of the pulse is to be described or predicted in detail. We study now a particularly simple and useful model of time-dependent light fields based on classical diffusion theory, in particular, equation (7) of Sec. 6.5.

Consider an infinite homogeneous optical medium with a single point source at x' which at time t' emits a single Dirac-delta pulse of unit radiant energy. That is, we assume h_η in (7) of Sec. 6.5 to have the form: $h_\eta(x,t) = U_\eta \delta(x-x') \cdot \delta(t-t')$, where at present $U_\eta = 1$, and U_η in general has the dimensions of radiant energy.

It may be verified directly from (7) of Sec. 6.5 (by performing the indicated differentiations and simplifying) that the resultant scalar irradiance $h(x,t)$, $t > t'$, varies in space and time according as $K_\kappa(x',x;t',t)$, where we have written:

$$\text{"}K_\kappa(x',x;t',t)\text{" for } \frac{v}{[4\pi v D(t-t')]^{3/2}} \exp\left\{ - \frac{|x-x'|^2}{4vD(t-t')} - av(t-t') \right\} \tag{82}$$

That is, for fixed x' and t', the function $K_\kappa(x',\cdot\,;t',\cdot)$ defined by (82) satisfies (7) of Sec. 6.5 at every space-time point (x,t), such that $x' \neq x$ and $t > t'$. The function $K_\kappa(x',\cdot\,;t',\cdot)$ first arose in the theory of transient heat conduction.

In general, with a continuous source distribution $h_\eta(x',t')$ defined throughout a part X_η of the medium for all times $t > t'$, we have, by means of the interaction principle, the resultant scalar irradiance field given by:

$$\boxed{h(x,t) = \int_{X_\eta} \int_{-\infty}^{t} h_\eta(x',t')\, K_\kappa(x',x;t',t)\, dt\, dV(x')} \tag{83}$$

Of course, h_η may be set equal to zero for all times t' earlier than some fiducial time t_o', so that $h_\eta(x',t')$ in (83) represents the general source condition (7) of Sec. 6.5. Therefore the resultant scalar irradiance field h defined by

(83) is the general solution of (7) of Sec. 6.5, as may be established by a direct appeal to (7) of Sec. 6.5.

It is of interest to connect (83) with two results obtained earlier in the present work. First we will show that if a steady point source condition subsists for all time, i.e., $h_\eta(x',t')$ is independent of time t' for all $t' < t$ and is zero for all points x other than a given point x' on the medium, then:

$$\boxed{K_\kappa(x',x) = \int_{-\infty}^{t} K_\kappa(x',x;t',t)\, dt'} \tag{84}$$

so that (83) reduces to the steady state case (54). To see this we note that $K_\kappa(x',x;t',t)$ has the general Gestalt of:

$$a\,\frac{e^{\left(-\frac{b}{t} - ct\right)}}{t^{3/2}}$$

where we have written, *ad hoc*:

$$\text{"a"} \quad \text{for} \quad \frac{v}{[4\pi vD]^{3/2}}$$

$$\text{"b"} \quad \text{for} \quad \frac{|x-x'|^2}{4vD}$$

and:

$$\text{"c"} \quad \text{for} \quad av$$

and have replaced occurrences of "(t-t')" by "t". Then it is clear that on setting $t = u^2$:

$$\int_{-\infty}^{t} K_\kappa(x',x;t',t)\, dt' = 2a\int_0^\infty \frac{e^{\left(-\frac{b}{u^2} - cu^2\right)}}{u^2}\, du$$

$$= \frac{a\sqrt{\pi}}{\sqrt{b}}\, e^{-2\sqrt{bc}}$$

$$= \frac{e^{-\kappa|x-x'|}}{4\pi D|x-x'|} = K_\kappa(x',x) \quad .$$

The second connection we can make is that between (83) and the earlier result which describes the behavior of radiant energy under standard decay conditions, namely, property

8 of Sec. 5.10. To establish this connection we now assume that $h_\eta(x,t) = U_\eta\ \delta(x)\ \delta(t)$. This simulates the instantaneous localized introduction of an amount U_η of radiant energy into the medium. However, the actual manner of introduction is immaterial for the present discussion. With this condition on h_η, (83) yields:

$$h(x,t) = U_\eta\ K_\kappa(0,x;0,t) \quad ,$$

so that the radiant energy content of the medium at time t is:

$$U(t) = \frac{1}{v}\int_X h(x,t)\ dV(x)$$

$$= \frac{U_\eta}{v}\int_X K_\kappa(0,x;0,t)\ dV(x)$$

$$= \frac{U_\eta\ e^{-avt}}{[4\pi vDt]^{3/2}}\int_X \exp\left\{-\frac{|x|^2}{4vDt}\right\}\ dV(x)$$

Hence:

$$U(t) = U_\eta\ e^{-avt} \quad ,$$

which is precisely the analytic content of property 8 of Sec. 5.10. This most interesting result shows that the classical diffusion theory is globally exact and thereby may be used to help fill, in a consistent manner, the general gap in our knowledge about the *local* radiance distributions within a time-dependent radiant field. That is, we may use (83) to supplement the exact theory of the time-dependent radiant energy field studied in Chapter 5, by giving approximate but useful estimates of the radiant density throughout the medium.

To implement the program just outlined of supplementing the exact radiant energy theory of Chapter 5 by diffusion theory, we construct the basic diffusion equations for n-ary scalar irradiance from the time-dependent equation of transfer (19) of Sec. 5.8. Thus, by applying the operator **U** to the equation of transfer for n-ary radiance, we have for $n \geq 1$:

$$\boxed{\frac{1}{v}\frac{\partial h^n}{\partial t} + \nabla \cdot \mathbf{H}^n = -\alpha h^n + s h^{n-1}} \qquad (85)$$

where for every $n \geq 1$ we have written:

$$\text{"}\mathbf{H}^n\text{"} \quad \text{for} \quad \int_{\Xi} N^n \xi \, d\Omega(\xi)$$

and: (86)

$$\text{"}h_*^n\text{"} \quad \text{for} \quad \int_{\Xi} N_*^n \, d\Omega(\xi)$$

Assuming Fick's law holds between $\mathbf{H}^n$ and h^n, for every n, $n \geq 1$, i.e., assuming:

$$\mathbf{H}^n = -D \nabla h^n \quad , \tag{87}$$

then (85) yields the time-dependent diffusion equation for n-ary scalar irradiance, $n \geq 1$:

$$\boxed{\frac{1}{v}\frac{\partial h^n}{\partial t} - D \nabla^2 h^n = -\alpha h^n + s h^{n-1}} \tag{88}$$

One immediate application of (88) is the direct generalization, to the time-dependent setting, of the results (68) and (79) of the continuous source cases with all the analytic advantages of those results now transferred to the time-dependent context. In particular, we can replace $h_\eta(x',t;)$ in (83) by $h_*^1(x',t')$ which is computed exactly as in (67), but with suitable time lag to account for the travel of the initial pulse of the source from the source to x'. Then we compute $h^*(x,t)$ as follows:

$$h^*(x,t) = \int_{X_\eta} \int_{-\infty}^{t} h_*^1(x',t') \, K_\kappa(x',x;t',t) \, dt' \, dV(x') \tag{89}$$

so that:

$$h(x,t) = h^o(x,t) + h^*(x,t) \tag{90}$$

where $h^o(x,t)$ is the residual scalar irradiance computed from the given source condition, which may be discrete or finite.

The theoretical basis for (89) is the time-dependent counterpart to (66). This time-dependent counterpart is obtained, e.g., by adding up all equations in (88) for $n = 1, 2, \ldots$.

The result is:

$$\boxed{\frac{1}{v}\frac{\partial h^*}{\partial t} - D\,\nabla^2 h^* = -\,ah^* + h^1_*} \tag{91}$$

Observe how the infinite number of Fick's laws in (87) imply (65). On the basis of (91), the representation (89) is established by simply repeating the arguments leading to (83). Finally, the generalization of (91) to the time-dependent version of (78), and the derivation of the corresponding representation of (79), is readily made following the patterns of derivation established in that steady-state case.

6.7 Solutions of the Exact Diffusion Equations

The exact diffusion equation on which we base the discussion of the present section is (57) of Sec. 6.5. In full notation, this equation is of the form:

$$h(x) = \frac{1}{4\pi}(h_\eta + sh)\,\mathbf{V}(x)$$

$$= \frac{1}{4\pi}\int_X \Big(h_\eta(x') + s(x')\,h(x')\Big)\,K_\alpha(x',x)\,dV(x')$$

$$= \frac{1}{4\pi}\int_X \Big(h_\eta(x') + s(x')\,h(x')\Big)\,\frac{T_{r-r'}(x',\xi)}{|x-x'|^2}\,dV(x') \tag{1}$$

The current settings in which this integral equation is to describe the scalar irradiance field h are infinite and semi-infinite homogeneous media with arbitrary sources described by h_η within X. Once a solution h is found for a space X, the associated radiance distribution throughout X is obtained by means of (60) of Sec. 6.5. The first of our two main goals in this section is to solve (1) for a point source in an infinite medium and arrange the solution in such a manner as to be directly applicable to problems of finding radiance distributions associated with general source conditions in X. It will be seen that by judiciously tabulating the point source solution of (1), all solutions of (1) corresponding to the possible source conditions within X, are obtainable in principle by relatively straightforward numerical procedures based on the tabulated solution. The second main goal is to discuss the solutions of (1) for semi-infinite media (infinitely deep, plane-parallel media) with arbitrary internal sources.

Infinite Medium with Point Source

We begin with (1) for the case of an infinite homogeneous medium X with a point source at the origin. The homogeneity assumption frees $\alpha(x)$ and $s(x)$ of dependence on x throughout X and lets us write:

$$\frac{T_{r-r'}(x',\xi)}{|r-r'|^2} = \frac{e^{-\alpha|x-x'|}}{|x-x'|^2} \tag{2}$$

where, as usual "x" denotes a point in X, and where $|x-x'|$ is the distance between points x and x'. The point source condition is represented by:

$$h_\eta(x') = P_\eta \ \delta(x') \tag{3}$$

where P_η is the quantity of radiant flux emitted steadily in time and uniformly in all directions by the point source at the origin. We may leave the nature of this source quite arbitrary throughout the discussion. As a result, we shall be able to adapt various solutions of (1) for the point source case, by means of integration, and in such a manner that the actual nature of the source may vary from true emission processes, through transpectral scattering processes, on through elastic scattering processes. This will be illustrated later in the discussion. For the present we go on to investigate the case of (1) with a single point source. The requisite form of (1) is:

$$h(x) = \frac{1}{4\pi} \int_X \left[P_\eta \ \delta(x') + sh(x') \ \frac{e^{-\alpha|x-x'|}}{|x-x'|^2} \right] dV(x') \tag{4}$$

The theory of the solution of (4) is thoroughly understood; a representative detailed development of the solution of (4) may be found, e.g., in [40]. Therefore, beyond the general observations leading from (39) to (59) of Sec. 6.5, we shall not need to discuss the details of the solution procedure of (4) in the present work. However, we wish to display the solution of (4) in such a manner that the results of [40] may be readily adapted to the radiative transfer context. Such an adaptation requires the preliminary transition to a certain class of dimensionless geometric parameters, which we now define.

Throughout this section we shall write:

$$\text{"}\tau(x,x') \quad \text{for} \quad \int_0^r \alpha(x'') \, dr\text{"} \tag{5}$$

where α is the volume attenuation function for the medium. The integral is a line integral along a path $\mathcal{P}_T(x,\xi)$ with initial point x and terminal point x'. Since the medium X is isotropic and homogeneous, paths are straight-line segments and

$$\tau(x,x') = \alpha|x-x'| \tag{6}$$

When no confusion will result, we will simply write:

$$\text{"}\tau\text{"} \quad \text{for} \quad \tau(x,x') \quad ,$$

with x, and x' thereby being understood.

The quantity τ assigned to the distance $|x-x'|$ between x and x' is dimensionless, and by virtue of (6) may be viewed as the number of attenuation lengths L_α between x and x'.

Next, for every subset Y of X we write:

$$\text{"}V_\alpha(Y)\text{"} \quad \text{for} \quad \int_Y \alpha^3(x')\, dV(x') \tag{7}$$

The quantity $V_\alpha(Y)$ is dimensionless. Throughout this section, both $\tau(x,x')$ and V (Y) may be thought of and referred to as *optical lengths* and *optical volumes*, respectively, without fear of confusion with the classical notions of the same names.

With definitions (5) and (7) in mind, (4) may be rewritten as:

$$h(x) = \frac{1}{4\pi} \int_X \left[\frac{P_\eta}{\alpha}\, \delta(x') + \rho\, h(x') \; \frac{e^{-\tau(x,x')}}{\tau^2(x,x')} \right] dV_\alpha(x') \tag{8}$$

where ρ is the scattering-attenuation ratio s/α. Equation (8) is the required dimensionless version of (4); and for purposes of a solution tabulation, we now impose the ***unit source condition*** in the context of (8):

$$\frac{P_\eta}{\alpha} = 1 \tag{9}$$

provided that the Dirac-delta function δ with dimensions L^{-3} (to go with the volume measure V) is retained. Otherwise, if a dimensionless Dirac-delta function δ (to go with the optical V_α) is adopted, in (3) we write $h_\eta\, \delta(x')$ and the unit source condition is

$$\frac{h_\eta}{\alpha} = 1 \quad . \tag{9a}$$

The scalar irradiance field h governed by (8) is clearly spherically symmetric about the point source so that h depends only on radial distance r or (now that the transition to dimensionless parameters has been made) on τ. Let us denote the solution of (8), *under the unit source condition (9a)*, by "K_ε". Then it can be shown (cf. [40]) that the scalar irradiance at optical distance τ from the origin is $K_\varepsilon(\tau)$, where:

$$\boxed{K_\varepsilon(\tau) = A(\rho,\tau)\, K_\alpha(\tau) + B(\rho,\tau)\, K_\kappa(\tau)} \tag{10}$$

and where, in turn we have written:

$$\text{"}A(\rho,\tau)\text{"} \quad \text{for} \quad \frac{1}{4\pi}\, \varepsilon(\rho,\tau) \tag{11}$$

and

$$\text{"}B(\rho,\tau)\text{"} \quad \text{for} \quad D_o \frac{\partial k_o^2}{\partial \rho} \tag{12}$$

to point up the fact that $K_\varepsilon(\tau)$ is simply a linear combination of the dimensionless diffusion kernel $K_\kappa(\tau)$ (cf. (52) of Sec. 6.6) where now we write:

$$\text{"}K_\kappa(\tau)\text{"} \quad \text{for} \quad \frac{e^{-\kappa_o\tau}}{4\pi D_o \tau} \tag{13}$$

and the dimensionless beam transmittance kernel $K_\alpha(\tau)$ (cf. (43) of Sec. 6.5) where now we write:

$$\text{"}K_\alpha(\tau)\text{"} \quad \text{for} \quad \frac{e^{-\tau}}{\tau^2} \quad . \tag{14}$$

It remains to specify the terms $\varepsilon(\rho,\tau)$, κ_o, $\partial k_o^2/\partial\rho$, and D_o. The latter term is simply αD, where D is the diffusion constant (cf. (27) of Sec. 6.5) for the classical diffusion theory. The remaining three terms form the heart of the exact solution and are tabulated in Tables 1 and 2 below for various values of ρ and τ.

Thus from (10), we have

$$\boxed{K_\varepsilon(\tau) = \frac{\varepsilon(\rho,\tau)}{4\pi\tau^2}\, e^{-\tau} + \frac{\partial k_0^2}{\partial\rho}\, \frac{1}{4\pi\tau}\, e^{-\kappa_0\tau}} \tag{15}$$

TABLE 1

The function $\varepsilon(\rho,\tau)$

τ	$\rho = 0$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0000	1.0210	1.0418	1.0542	1.0526	1.0420
0.2	1.0000	1.0382	1.0773	1.1000	1.0962	1.0756
0.3	1.0000	1.0532	1.1088	1.1409	1.1346	1.1046
0.4	1.0000	1.0667	1.1375	1.1781	1.1692	1.1301
0.5	1.0000	1.0790	1.1640	1.2126	1.2008	1.1529
0.6	1.0000	1.0904	1.1888	1.2448	1.2300	1.1736
0.7	1.0000	1.1010	1.2121	1.2752	1.2571	1.1926
0.8	1.0000	1.1109	1.2342	1.3038	1.2826	1.2100
0.9	1.0000	1.1202	1.2552	1.3311	1.3066	1.2262
1.0	1.0000	1.1291	1.2753	1.3571	1.3293	1.2412
1.5	1.0000	1.1674	1.3644	1.4724	1.4273	1.3034
2.0	1.0000	1.1990	1.4402	1.5699	1.5068	1.3504
2.5	1.0000	1.2258	1.5068	1.6551	1.5738	1.3874
3.0	1.0000	1.2494	1.5667	1.7311	1.6314	1.4171
3.5	1.0000	1.2704	1.6213	1.8000	1.6818	1.4415
4.0	1.0000	1.2895	1.6718	1.8630	1.7265	1.4617
4.5	1.0000	1.3070	1.7188	1.9214	1.7665	1.4786
5.0	1.0000	1.3231	1.7630	1.9757	1.8026	1.4928
6.0	1.0000	1.3521	1.8443	2.0745	1.8654	1.5147
7.0	1.0000	1.3779	1.9182	2.1630	1.9182	1.5304
8.0	1.0000	1.4010	1.9863	2.2432	1.9634	1.5412
9.0	1.0000	1.4222	2.0497	2.3169	2.0024	1.5486
10.0	1.0000	1.4417	2.1094	2.3851	2.0366	1.5531
11.0	1.0000	1.4599	2.1659	2.4499	2.0667	1.5554
12.0	1.0000	1.4770	2.2196	2.5086	2.0933	1.5559
13.0	1.0000	1.4931	2.2710	2.5652	2.1172	1.5550
14.0	1.0000	1.5084	2.3204	2.6188	2.1385	1.5529
15.0	1.0000	1.5230	2.3682	2.6700	2.1578	1.5498
16.0	1.0000	1.5370	2.4141	2.7190	2.1752	1.5459
17.0	1.0000	1.5503	2.4586	2.7658	2.1910	1.5413
18.0	1.0000	1.5632	2.5019	2.8109	2.2055	1.5361
19.0	1.0000	1.5757	2.5439	2.8543	2.2186	1.5304
20.0	1.0000	1.5877	2.5849	2.8963	2.2307	1.5243

Now that it is clear how $K_\varepsilon(\tau)$ depends on the diffusion kernel K_κ ((52) of Sec. 6.6) and the attenuation kernel K_α ((43) of Sec. 6.5) we write (10) in its explicit form:

TABLE 1--Concluded

The function $\varepsilon(\rho,\tau)$.

τ	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 1.0$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0269	1.0099	0.9921	0.9745	0.9564
0.2	1.0474	1.0162	0.9843	0.9528	0.9222
0.3	1.0643	1.0206	0.9767	0.9341	0.8934
0.4	1.0786	1.0236	0.9693	0.9173	0.8683
0.5	1.0909	1.0257	0.9621	0.9019	0.8460
0.6	1.1017	1.0271	0.9551	0.8878	0.8260
0.7	1.1113	1.0279	1.9483	0.8747	0.8077
0.8	1.1198	1.0282	0.9417	0.8625	0.7910
0.9	1.1275	1.0282	0.9353	0.8510	0.7755
1.0	1.1343	1.0278	1.9290	0.8402	0.7612
1.5	1.1601	1.0229	0.9002	0.7936	0.7019
2.0	1.1763	1.0149	0.8748	0.7562	0.6568
2.5	1.1866	0.0054	0.8519	0.7250	0.6207
3.0	1.1929	0.9952	0.8313	0.6982	0.5908
3.5	1.1963	0.9847	0.8124	0.6749	0.5655
4.0	1.1978	0.9742	0.7951	0.6543	0.5437
4.5	1.1976	0.9637	0.7791	0.6358	0.5246
5.0	1.1963	0.9534	0.7643	0.6191	0.5076
6.0	1.1912	0.9334	0.7374	0.5901	0.4788
7.0	1.1838	0.9144	0.7137	0.5654	0.4550
8.0	1.1749	0.8964	0.6926	0.5440	0.4349
9.0	1.1651	1.8793	0.6734	0.5253	0.4175
10.0	1.1547	0.8631	0.6560	0.5086	0.4024
11.0	1.1438	0.8477	0.6400	0.4936	0.3890
12.0	1.1327	0.8330	0.6252	0.4800	0.3769
13.0	1.1215	0.8190	0.6114	0.4676	0.3661
14.0	1.1102	0.8055	0.5985	0.4562	0.3562
15.0	1.0989	0.7926	0.5864	0.4456	0.3471
16.0	1.0876	0.7802	0.5750	0.4357	0.3387
17.0	1.0764	0.7683	0.5643	0.4265	0.3310
18.0	1.0653	0.7568	0.5540	0.4178	0.3238
19.0	1.0542	0.7457	0.5443	0.4096	0.3170
20.0	1.0433	0.7349	0.5349	0.4019	0.3107

TABLE 2

The functions κ_0 and $d\kappa_0^2/d\rho$

ρ	κ_0	$d\kappa_0^2/d\rho$
0.0	1.000000	0.000000
0.1	1.000000	0.164892(-5)*
0.2	0.999909	0.009094
0.3	0.997414	0.116201
0.4	0.985624	0.373272
0.5	0.957504	0.731896
0.6	0.907332	1.145954
0.7	0.828635	1.590033
0.8	0.710412	2.051119
0.9	0.525430	2.522370
0.92	0.474002	2.617473
0.94	0.413976	2.712805
0.96	0.340829	2.808348
0.98	0.242983	2.904085
0.99	0.172511	2.952020
1.00	0.000000	3.000000

*Note: "(-5)" means "multiply by 10^{-5}."

In this way we can see that, for computation purposes, the scalar irradiance $K_\varepsilon(\tau)$ at optical distance τ from the origin consists of two terms, one which may be attributed to residual flux (the first term) and the other which may be attributed to scattered flux. This type of partitioning of the exact representation of h(x) into a residual part (h^o) and a scattered part (h^*) was already encountered in the classical diffusion theory, e.g., in (7) of Sec. 1.5, in (57) of Sec. 6.6, and more generally in (79) of Sec. 6.6. Also, in the time-dependent case, this partition was encountered in (90) of Sec. 6.6.

A tabulation of $4\pi\,\tau^2 K_\varepsilon(\tau)$ is given in Table 3 for two cases of ρ and for a range of τ from 0 to 10 units. These choices of ρ are representative orders of magnitude for ρ in the case of the ocean (ρ = 0.3) and the atmosphere (ρ = 0.9) for wavelengths around 500 mμ, for the middle of the visible spectrum. For the determination of $K_\varepsilon(\tau)$ for values of ρ other than ρ = 0.3, 0.9, Tables 1 and 2 may be used. It must be kept in mind that these tabulations are for the unit source condition (9a).

TABLE 3

The function $4\pi \tau^2 K_\varepsilon(\tau)$

τ	$\rho = 0.3$	$\rho = 0.9$
0.0	1.0000	1.0000
0.1	0.9644	1.1211
0.2	0.9196	1.2343
0.3	0.8710	1.3384
0.4	0.8209	1.4326
0.5	0.7708	1.5168
0.6	0.7215	1.5914
0.7	0.6737	1.6567
0.8	0.6277	1.7130
0.9	0.5838	1.7607
1.0	0.5421	1.8006
1.5	0.3675	1.8974
2.0	0.2441	1.8660
2.5	0.1599	1.7547
3.0	0.1037	1.5992
3.5	0.0668	1.4239
4.0	0.0427	1.2454
4.5	0.0272	1.0742
5.0	0.0173	0.9158
6.0	0.0069	0.6483
7.0	0.0028	0.4467
8.0	0.0011	0.3018
9.0	0.0004	0.2007
10.0	0.0002	0.1318

Infinite Medium with Arbitrary Sources

We now develop a procedure whereby Table 3, and more generally (15), may be used to compute scalar irradiance fields generated by arbitrary sources. Suppose the source term $h_\eta(x)$ is given throughout an infinite medium X; $h_\eta(x)$ may be associated with plane sources, finite volume sources of flux, etc., and may be of quite arbitrary spatial dependence throughout X. It is clear either intuitively or formally (from the interaction principle using the theorems of Sec. 3.16) that the scalar irradiance h(x) associated with $h_\eta(x)$ is given by:

$$\boxed{h(x) = \frac{1}{\alpha} \int_X h_\eta(x')\ K_\varepsilon(x',x)\ dV_\alpha(x')} \tag{16}$$

where we have written:

$$\text{"}K_\varepsilon(x',x)\text{"} \quad \text{for} \quad K_\varepsilon(\tau(x,x')) \tag{17}$$

The reason for the presence of "α" in (16) may be found by tracing back through the unit source condition (9a) and ultimately to (3) and (4). If h_η is given in watts per cubic meter, and α in per meter, then h is given in units of watts per square meter.

A practical computation scheme for h(x) may be based on the following procedure: given $h_\eta(x)$ throughout a subset X_η of X, divide X_η into n small cubes $C(x_i)$ (or any other conveniently shaped regions) over each of which both $\tau(x,x')$ and $h_\eta(x)$ vary only slightly. Thus each cube $C(x_i)$ is representative of the radiometric properties of X around x_i, where x_i is the cube's centerpoint. Then (16) may be replaced by the approximating finite sum:

$$\boxed{h(x) = \frac{1}{\alpha} \sum_{i=1}^{n} h_\eta(x_i)\ K_\varepsilon(x_i,x)\ V_\alpha(C(x_i))} \tag{18}$$

The evaluation of h(x) using (18) is facilitated by using Table 3 for optical distances $\tau(x,x')$ up to 10. More generally, (15) would be used with Tables 1 and 2.

As a specific example of a setting in which (18) may be applied, consider the problem of determining the irradiance field generated in an infinite homogeneous medium by a beam-type source, such as that associated with powerful search lights or laser beams. The geometrical relations of the present example are summarized in Fig. 6.5. The source may be represented as a small sphere of radius r_o with surface radiance N_o and which is allowed to emit uniformly over a conical set Ξ_o of directions with central direction ξ_o. Thus Ξ_o may be all directions ξ such that $\xi \cdot \xi_o \geq \cos\theta_o$ where θ_o is the half angle opening of Ξ_o. By varying θ_o, the cone can represent everything from narrow beams (small θ_o) to uniform point sources ($\theta_o = \pi$).

With these geometrical preliminaries fixed, we now return to the discussion in Sec. 6.6 which developed the theory of primary scattered flux as source flux and which culminated in the formulas (67) through (69) of Sec. 6.6. We can

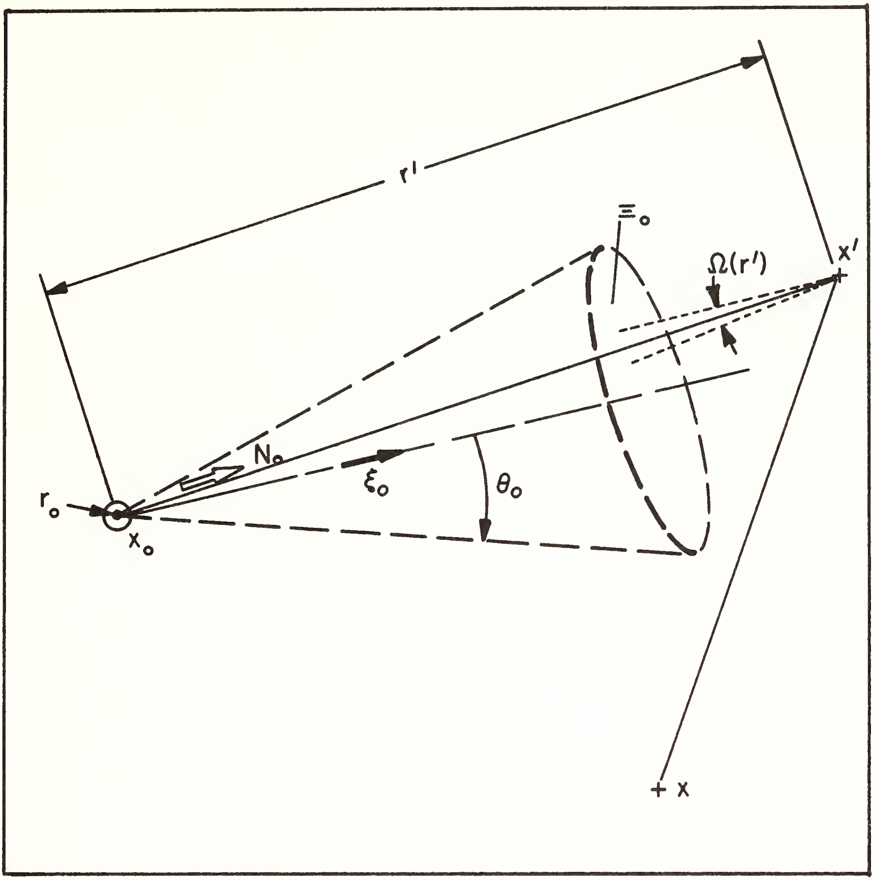

FIG. 6.5 Geometry for a nonisotropic point source of radiant flux in diffusion theory.

immediately adopt for our present purposes the formula (67) of Sec. 6.6 which describes the primary scalar irradiance $h_*^1(x')$ in terms of the inherent radiance N_o, the total scattering coefficient s, the beam transmittance $e^{-\alpha r'}$, and the solid angle $\Omega(r')$ subtended by the point source at point x'. (See Fig. 6.5.) Now $h_*^1(x')$ replaces $h_\eta(x')$ in (16) or $h_\eta(x_i)$ in (18). Thus (16) becomes:

$$h(x) = \rho N_o \int_{X_o} e^{-\alpha|x'|}\, \Omega(|x'|)\, K_\varepsilon(x',x)\, dV_\alpha(x') \qquad (19)$$

and (18) becomes:

$$h(x) = \rho N_o \sum_{i=1}^{n} e^{-\alpha|x_i|}\ \Omega(|x_i|)\ K_\varepsilon(x_i,x)\ V_\alpha(C(x_i)) \qquad (20)$$

In (19) the integration may be limited to the subset X_o of X defined by the cone Ξ_o of directions. Thus point x' is in X_o if and only if $x'/|x'|$ is in Ξ_o. In (20) the sum is over all cells $C(x_i)$ which partition X_o. Because of the exponentials and the solid angles $\Omega(|x_i|)$ in (20), the sums (for a given N_o) need not be extended over very many attenuation lengths within X_o before good estimates of $h(x)$ can be made.

Semi-Infinite Medium with Boundary Point Source

The exact diffusion solution (16) holds for media which extend indefinitely far in all directions about the point source. Such a situation will hold more or less in natural waters when the source and observer are at relatively great depths (several attenuation lengths, say). However, if the source is relatively near the surface, the reflectance properties of the remaining thin layer of medium above the source would differ noticeably from that of an infinitely deep layer above the source, so that the scalar irradiance $h(\tau)$ at shallow depths in a light field induced by a point source near the boundary would differ markedly from that predicted by (16). Similar observations may be made for fogs and cloud banks in the atmosphere. In the present example, we summarize some results of exact diffusion theory which can predict $h(\tau)$ for relatively shallow depths in natural waters (or for points near flat cloud or fog boundaries) when the point source is on the boundary. The reflection effects of the air-water surface are not included in the present analysis and must be accounted for separately. In the second example below the results will be extended to the case of internal point sources. Both examples are based on the results by Elliott given in Ref. [88]. A generalization of the equations developed below and their appropriate place in the general theory of radiative transfer in media with internal sources, will be given in Sec. 7.13.

The starting point for the present discussion is equation (8) in which the medium X is now an infinitely deep homogeneous plane-parallel medium exhibiting isotropic scattering and with a point source of small positive radius r_o at depth $x = c \geq 0$. We shall use the terrestrially based reference system for natural hydrosols (cf. Sec. 2.4). Furthermore we use the unit source condition (9a) in (8).

Thus we start with (8), now in the form:

$$h(x) = \frac{1}{4\pi} \int_{X_+} (\delta(x'-x_o) + \rho h(x'))\ \frac{e^{-\tau(x,x')}}{\tau^2(x,x')}\ dV_\alpha(x') \qquad (21)$$

where X_+ is the set of all x $(=(x_1,x_2,x_3))$ in the terrestrial coordinate frame such that $x_3 = z \geq 0$. The Dirac-delta function ξ in (21) is dimensionless, and is centered on the point x_0 $(=(0,0,c))$, $c \geq 0$. Furthermore, it is to be explicitly noted that for the remainder of this section all coordinates x_1, x_2, x_3 (hence all distances, areas, and volumes) are to be measured in units of optical length (cf. (5), (7)).

Now the procedure in Ref. [88] is to take the Fourier transform of (21) with respect to the variables x_1 and x_2 over an arbitrary horizontal plane at depth x_3 $(=z)$. Thus let ω_1 and ω_2 be the spatial frequencies along the x_1 and x_2 directions and let us write:*

$$\text{"}f_o(z;\omega_1,\omega_2)\text{"} \quad \text{for} \quad \int_{X_z} h(x)\, e^{i(x_1\omega_1 + x_2\omega_2)}\, dA(x) \tag{22}$$

where X_z is the horizontal plane at depth z, and A is the area measure over X_z. Thus f_o is the Fourier transform of h over X_z, and f_o has the same dimensions as h. Therefore, applying the operator:

$$\int_{X_z} [\quad]\, e^{i(x_1\omega_1 + x_2\omega_2)}\, dA(x)$$

to each side of (21), we obtain:

$$f_o(z;\omega) = \frac{\rho}{2} \int_0^\infty \left[\frac{1}{\rho}\,\delta(z'-a) + f_o(z';\omega)\right] I(|z-z'|,\omega)\, dz' \tag{23}$$

where we have written:

$$\text{"}I(|z-z'|,\omega)\text{"} \quad \text{for} \quad \int_1^\infty \frac{e^{-|z-z'|t}}{t}\, J_o\left(\sqrt{\omega_1^2+\omega_2^2}\; |z-z'|\; \sqrt{t^2-1}\right) dt \tag{24}$$

---

*In the present exposition, we retain the Fourier transform conventions used in [88] in order to facilitate the study of the results therein.

where J_0 is a zero-order Bessel function, and where, for brevity, we have written:

$$\text{"}f_o(z;\omega)\text{"} \quad \text{for} \quad f_o(z,\omega_1,\omega_2) \tag{25}$$

The next step in the solution procedure is the observation that (23) can be solved using the Wiener-Hopf technique provided that $c = 0$, i.e., that the source is at the boundary. This solution procedure is quite intricate and beyond the immediate interests of the present work; therefore the interested reader is referred to Ref. [88] for details and further references. The main results of the present example may be understood without recourse to the solution details. We need only observe that the required scalar irradiance is obtained from the solution $f_o(\cdot;\omega)$ of (23) by means of the following integration which is the inverse Fourier transformation to that in (22):

$$h(x) = \frac{1}{2\pi} \int_0^\infty f_o(z,\omega)\ \omega\ J_o(\omega r)\ d\omega \tag{26}$$

in which:

$$x = (x_1, x_2, z)$$

and:

$$\omega^2 = \omega_1^2 + \omega_2^2 \tag{27}$$

$$r^2 = x_1^2 + x_2^2 \tag{28}$$

Since $h(x)$ depends only on depth z and the radial distance r, we agree to write:

$$\text{"}h(z,r)\text{"} \quad \text{for} \quad h(x) \tag{29}$$

Figure 6.6 depicts the geometrical details of the case where the point source is at the boundary. Observe that the medium is divided into region A (shaded) and conical region B (unshaded). It is found that $h(z,r)$ for points $x = (x_1,x_2,z)$ in region A is approximated by the relation:

$$\boxed{h(z,r) = \frac{\sqrt{3}\ h_\eta\ \psi_1(z)}{2\pi\alpha r^3}\ e^{-\kappa_o r}(1 + \kappa_o r)} \tag{30}$$

(Valid in region A, Fig. 6.6.)

where in turn $\psi_1(z)$ is evaluated in [172] and is tabulated in Table 4, and κ_0 is given in Table 2. Table 4 may be extended,

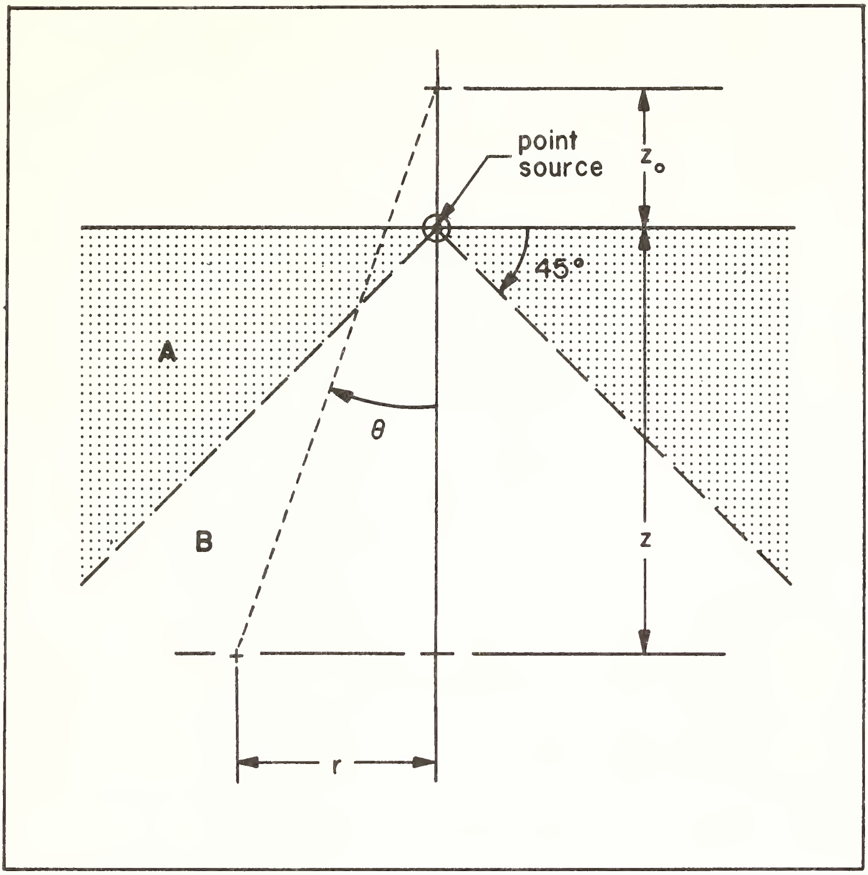

FIG. 6.6 Domains of validity of approximate solutions (30) and (31).

if necessary, using the eddingtonian approximation to ψ_1:

$$\psi_1(z) = \left(z + \frac{17}{24}\right)\left[\frac{2 + 3z - (E_2(z) - \frac{3}{2} E_3(z))}{2 + 3z - 3\,(E_4(z) - \frac{3}{2} E_5(z))}\right]$$

The functions $E_n(z)$ are the exponential integrals

$$\int_1^\infty u^{-n} e^{-zu}\, du \quad ,$$

and are tabulated. The farther the point x $(=(x_1,x_2,x_3))$ in region A is from the dashed dividing lines between regions A and B, the better the approximation (30).

TABLE 4

Evaluation of $\psi_1(z)$

z	$z + z_o$	$\psi_1(z)$
0	0.7104	0.5773
0.01	0.7204	0.5982
0.02	0.7304	0.6154
0.03	0.7404	0.6312
0.05	0.7604	0.6607
0.1	0.8104	0.7279
0.2	0.9104	0.8495
0.3	1.0104	0.9633
0.4	1.1104	1.0731
0.5	1.2104	1.1803
0.6	1.3104	1.2858
0.7	1.4104	1.3901
0.8	1.5104	1.4935
0.9	1.6104	1.5963
1.0	1.7104	1.6985
1.2	1.9104	1.9019
1.5	2.2104	2.2051
2.0	2.7104	2.7079
2.5	3.2104	3.2092
3.0	3.7104	3.7098
3.5	4.2104	4.2101
4.0	4.7104	4.7102

The error of the approximation by (30) is of the order of magnitude of $|z^3/r^5|$ and (30) is applicable when ρ is 0.6 or more.

Furthermore, it is found that $h(z,r)$ for points x $(=(x_1, x_2, z))$ in region B is approximated by the relation:

$$\boxed{h(z,r) = \frac{\sqrt{3}\, h_\eta \cos\theta}{2\pi\alpha d^2} e^{-\kappa_o d}(1 + \kappa_o d) \quad \text{(Valid in region B, Fig. 6.6.)}} \tag{31}$$

where we have written:

$$\text{"d" for } \sqrt{r^2 + (z+z_o)^2} \tag{32}$$

and where:

$$\tan\theta = \frac{r}{z + z_o} \tag{33}$$

and:

$$z_o = 0.7104 \tag{34}$$

This approximation improves with the distance of x ($=x_1,x_2,z$)) in region B from the dashed dividing lines between regions A and B. The error of approximation by (31) is of the order of magnitude of $|1/d^5|$ and (31) is applicable when ρ is 0.6 or more.

A study of (30) and (31) readily shows the effect on h(x) of the presence of the boundary at depth $z = 0$. Suppose for the moment that $\kappa_o = 0$ (no absorption case). Then in region A of Fig. 6.6, and for fixed z, the scalar irradiance falls off as the inverse cube of the distance r from the symmetry axis of the field, whereas in region B, which is relatively farther removed from the boundary than region A, the scalar irradiance falls off only as the inverse square of the distance d. The fixed number z_o (known as the "extrapolation length") in (34) arises in the correct adjustment of boundary conditions of the present problem.

Semi-Infinite Medium with Internal Point Source

The results of the preceding example will now be extended to the case of a semi-infinite homogeneous medium with a point source at $x_o = (0,0,c)$, $c \geq 0$, i.e., with a point source in the interior of the medium rather than on the boundary. Let us denote the solution of (21) for this case by "$f_c(z;\omega)$". Hence, when $c = 0$ we are to have $f_o(z;\omega)$ of (23) back once again, and f_c is to be a proper generalization of f_o. Now assume a general point source condition h_η/α (cf. (9a)). Then the functional relations connecting f_c and f_o, as derived by Elliott [88] are of the form:

$$f_c(z,\omega) = f_o(|z-c|,\omega) + \frac{s}{h_\eta}\int_0^z f_o(t,\omega)\, f_o(t+c-z,\omega)\, dt, \quad , \quad z \leq c \tag{35}$$

$$f_c(z,\omega) = f_o(|z-c|,\omega) + \frac{s}{h_\eta}\int_0^c f_o(t,\omega)\, f_o(t-c+z,\omega)\, dt \quad , \quad z \geq c \tag{36}$$

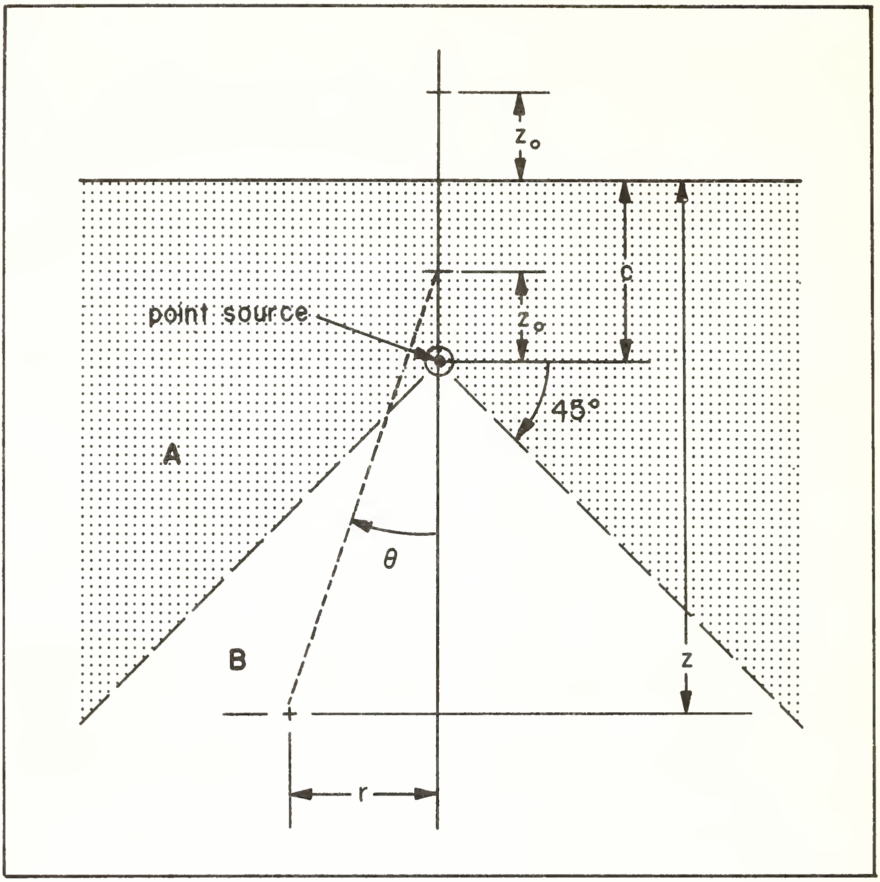

FIG. 6.7 Domains of validity of approximate solutions (38) through (40).

Once $f_c(z,\omega)$ is obtained using (35) or (36), $h(z,r)$ can be obtained by means of the inversion formula:

$$h(z,r) = \frac{1}{2\pi} \int_0^\infty f_c(z,\omega)\, \omega J_o(\omega r)\, d\omega \tag{37}$$

which is simply (26) now with f_c in place of f_o. A few observations on these functional relations will be made below, but for the present we go on to their immediate consequences. Figure 6.7 depicts the semi-infinite medium with point source at (0,0,c). The medium is divided into two regions with the shaded region A and the conical region B, exactly analogously to the partition depicted in Fig. 6.6. Corresponding to (30) we now have the approximate solution:

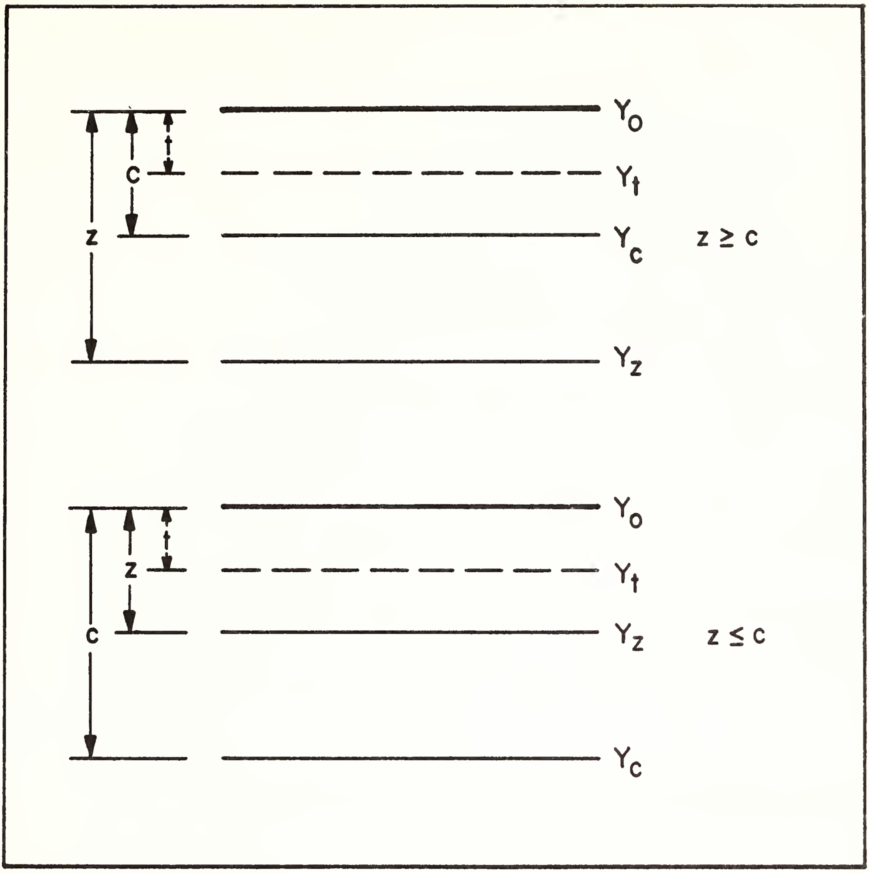

FIG. 6.8 Relative placement of source (c) and observation (z) levels in (35) and (36).

$$h(z,r) = \frac{\sqrt{3}h_n}{2\pi\alpha r^3}\left[\psi_1(c-z) + \sqrt{3}\int_0^z \left(\psi_1(t) + \psi_1(t+c-z)\right) dt\right] e^{-\kappa_0 r}(1+\kappa_0 r) ,$$

(3

for $z \leq c$.

$$h(z,r) = \frac{\sqrt{3}h_n}{2\pi\alpha r^3}\left[\psi_1(z-c) + \sqrt{3}\int_0^c \left(\psi_1(t) + \psi_1(t+z-c)\right) dt\right] e^{-\kappa_0 r}(1+\kappa_0 r) ,$$

(3

for $z \geq c$.

(Valid in region A, Fig. 6.7.)

All the terms occurring in (38) and (39) were defined in (30). The ranges of integration may be visualized with the help of Fig. 6.8. Observe how (39) reduces to (30) when $c = 0$. The errors of approximation are on the order of $|c^3/r^5|$ for (38) and $|z^3/r^5|$ for (39). The approximations (38), (39) are applicable for media with $\rho = 0.6$ or more.

Corresponding to (31) we now have:

$$\boxed{h(z,r) = \frac{\sqrt{3}\,h_\eta}{2\pi\alpha d^2}\,(1 + c\sqrt{3})\cos\theta\, e^{-\kappa_o d}\,(1 + \kappa_o d)} \tag{40}$$

(Valid in region B, Fig. 6.7)

Observe in this instance, also, how (40) reduces to its limiting case (31) for $c = 0$, where now in (40) we have written:

$$\text{"d"} \quad \text{for} \quad \sqrt{r^2 + (z + z_o - c)^2} \tag{41}$$

and also where

$$\tan\theta = \frac{r}{z + z_o - c} \tag{42}$$

The approximation (40) holds for large $|z-c|$ and has an error on the order of magnitude of $|c/d^3|$, for media with $\rho = 0.6$ or more.

Observations on the Functional Relations for f_c and f_o

The various solutions displayed above for $h(z,r)$ in a semi-infinite medium are of great interest for two reasons. The first reason is clear enough: They supply additional information on the behavior of $h(x)$ in deep plane-parallel media in which there are point sources near the boundaries. The second reason for interest in these solutions does not exist so much on a practical level as on a theoretical or conceptual level. This interest centers on the *form of the functional relations* (35) and (36) which seem to hold considerable importance for radiative transfer theory. These two remarkable relations show how to connect the point source solution for the case $c = 0$ with that for the case $c > 0$. The general form of the functional relations (35) and (36) are those of the relations usually found by the techniques of invariant imbedding, the techniques growing out of the classical invariance principles of Chandrasekhar. It will be shown in Sec. 7.13 how the general counterparts of (35) and (36) for radiance fields may be deduced from the invariant imbedding relations (cf. also examples 2, 3, 5 of Sec. 3.9). As a result of the derivations in Sec. 7.13, there will be a

unified set of analytical techniques for solving internal-source problems in general optical media.

6.8 Bibliographic Notes for Chapter 6

The discussions of Sec. 6.1 leading to (36) of that section are based on some elementary properties of complete orthonormal families of functions, which in turn find their rightful place in Hilbert space theory, or general vector space theory. For an exposition of these ideas, see, e.g., [104]. The isolation of the two properties, namely: the *finite recurrence property* of the orthonormal family and the *isotropy property* of the medium led to the finite forms (26) of the abstract harmonic equations in Sec. 6.2. This explicit delineation of the necessary properties to be held jointly by orthonormal families and optical media, which lead to the abstract harmonic equations (26) of Sec. 6.2, appears to be new.

The exposition of the classical spherical harmonic method in Sec. 6.3 is based on that of Refs. [175] and [314]. The solution procedures of the classical spherical harmonic equations for plane-parallel media in Sec. 6.4 are based on modern algebraic methods in differential equation theory, such as those in [47]. Some innovations in numerical procedures in the spherical harmonic method may be found in [323] and [325]. The manner of approach to diffusion theory in Sec. 6.5 is dictated by the specific needs and outlook of geophysical radiative transfer theory. The classification of diffusion processes in Sec. 6.5 is of course only partially complete; a systematic investigation of such classified processes appears to be of some interest to radiative transfer theory, and offers interesting physically based problems in partial differential equation theory.

The general solutions of the classical diffusion equations in the opening paragraphs of Sec. 6.6 are widely known, useful formulas for scalar irradiance. The various primary scattered flux source methods and those based on higher ordered scattered flux sources in the latter part of Sec. 6.6 offer some novelty in the otherwise quite thoroughly formed classical method of treatment of the diffusion of light through scattering media. Furthermore, the particular needs of hydrologic optics and meteorologic optics has caused some emphasis to be placed on the representation of the radiance distribution $N(x,\cdot)$ throughout diffusing media. This resulted in derivations of formulas for $N(x,\xi)$ in general diffusion contexts, such as (29) of Sec. 6.5; and (14) and (40) of Sec. 6.6, which do not appear to be too widely known.

The solutions of the exact diffusion equations in Sec. 6.7 for the case of infinite media are based on the work in [40]. This work also contains many useful tables and graphs of associated solutions. The theory of semi-infinite media with point sources is relatively unexplored. However, reference [88] forms a definitive beginning of such a theory, and the latter half of the discussions in Sec. 6.7 are based on the results of [88].

Further References

Further references beyond those mentioned above and which contain contributions to the classical theory of transport phenomena may be briefly mentioned here. First of all there is the early definitive work by Hopf [111] on mathematical problems of radiative transfer in media which are in thermodynamic equilibrium. This work contains the germ of the modern operator theoretical approach to transfer problems which is continued in [37] and [143], and more recently in [251]. Another early definitive work on classical radiative transfer theory is that of Chandrasekhar [43] which develops a minor variant of the spherical harmonic method of the kind formulated by Wick in [319]. Applications of the Chandrasekhar theory are made by Lenoble in [108], [155], [156]. By far the most significant contribution in [43] is that of the principles of invariance, which were discussed in general in Chapter 3 above and which will be considered further in Chapter 7 below. The reference [62] also contains much useful mathematical information which is applicable to practical radiative transfer contexts. A relatively recent survey of radiative transfer theory and classical and exact diffusion theory may be found in [288].

Some tabulated solutions of the equation of transfer are given in [53], [91], and [11]. Diffusion theory from the point of view of Monte Carlo techniques is explored in [41] and [176]. Some recent numerical solutions for light fields in homogeneous slabs (with isotropic scattering) which blend the spherical harmonic method and the technique of invariant imbedding are given in [15] and [16].

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